Continuous Time Goes by* Russell

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Abstract

Russell and Walker proposed different ways of constructing instants from events. As an explanation of “time as a continuum”, Thomason favoured Walker’s construction. The present paper shows that Russell’s construction fares as well. To this end, a mathematical characterization problem is solved which corresponds to the characterization problem that Thomason solved with regard to Walker’s construction. It is shown how to characterize those event structures (formally: interval orders) which, through Russell’s construction of instants, become linear orders isomorphic to a given (or, deriving: to some–non-trivial ordered) real interval. As tools, separate characterizations for each of resulting (i) Dedekind completeness, (ii) separability, (iii) plurality of elements, (iv) existence of certain endpoints are provided. Denseness is characterized to replace Russell’s erroneous attempt. Somewhat minimal non-constructive principles needed are exhibited.

Keywords: time, Russell, instants from events, continuum, interval orders, axiom of choice.

*The song by H. Hupfeld, played in the movie ‘Casablanca’.
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1 Introduction and summary.

The problem the present paper is concerned with derives from Russell’s [42, Lecture IV] attempt mathematically to construct “instants” of time from “events”. Thomason’s [46] paper starts almost the same way. According to Thomason, however, Russell’s construction has the disadvantage that it is difficult to see what assumptions about the temporal relationships among events will ensure that the instants constructed comprise a continuum, isomorphic to the real numbers. Thomason [46] thus presents an alternative way—due to the physicist A. G. Walker1—of constructing instants of time out of events. Thomason shows of some conditions on the temporal relationships of events that they do ensure instants of time, as constructed

1For hints on Walker’s inheritance in physics, see footnote 16.
according to Walker, comprising a “continuum”. He then concludes that Walker’s theory offers, as Russell’s does not, a plausible explanation of time as a continuum.

The present paper shows (besides some by-products) how to single out those relations on sets of events which become order-isomorphic to the real numbers through Russell’s construction. More generally, it shows how to tell from such a relation whether the resulting order is in a class (closed under isomorphisms) of intervals of the real numbers—an interval being open in each, one, or no direction; ordered by the relation of being less. At least one of the characterizations presented might be considered a “refutation” of Thomason’s claim denying explanatory power of Russell’s theory. The keys to these characterizations are (i) a sufficient condition on events and their temporal relationships for Russellian time being dense, due to N. Wiener or to Russell; (ii) a characterization, due to Russell, of those events that get first or, resp., last instants by his construction. The sufficient condition according to (i) is modified here to obtain a necessary and sufficient one.

The main results are stated in Section 7. The sections which precede Section 7 merely explain the notions used in the latter. In proving claims afterwards, I keep books on what non-constructive choice or maximality principles (the very axiom of choice or something weaker) I use. It will even be shown that one of the characterizing conditions can be used as an alternative to any such principle in the relevant context. Some proofs merely replace existing Principia-notation proofs and thus may be helpful to at least some readers. The last section comments on some remaining aspects of continuumlikeness of time.

\footnote{Indeed, the conditions he presents are necessary as well. Kleinknecht [26] does something very similar, but presents a pair of conditions which is only sufficient, not necessary.}

\footnote{A real interval as defined below, \textit{i.e.}.

\footnote{This extends Thomason’s and Walker’s scope to something of which admittedly physicists will hardly acknowledge any use of—see Section 4 below.

\footnote{Subsection 9.4 will discuss this statement.—Maybe even Thomason’s “difficulty claim” is refuted.

\footnote{As defined in Corollary 1 below.

\footnote{I have had some troubles in distinguishing Russell’s from N. Wiener’s credits. According to the footnote of [49, p. 441], Wiener (the well-known mathematician who later founded “cybernetics”) investigated the matter on Russell’s suggestion. Indeed, at that time Wiener was a student under Russell at Cambridge University ([31, pp. 45ff.], [18]). Thus, while Wiener explicitly attributes the definition of “instants” under consideration here and another notion to Russell, it is no surprise when further credits are difficult to track. Moreover, the first edition of Russell’s [42] appeared in the same year as Wiener’s [49].}
2 Relations, linearity, and real intervals.

Already in explaining my goal I will come across several binary relations; so I declare my general conventions concerning them in advance.

When \( X, Y \) are any sets and \( R \subseteq X \times Y \) is some binary relation between them, I will write \( x R y \) instead of \( \langle x, y \rangle \in R \). Considering some number \( R_1, \ldots, R_n \) of binary relations, I will write \( x_1 R_1 x_2 R_2 x_3 \ldots x_n R_n x_{n+1} \) meaning that \( x_1 R_1 x_2, x_2 R_2 x_3, \ldots \) and \( x_n R_n x_{n+1} \).

A related set is an ordered pair \( \langle X, R \rangle \) where \( R \) is a binary relation on \( X \).

If \( R \) is no subset of \( X \times X \), \( \langle X, R \rangle \) is shorthand for \( \langle X, R \cap (X \times X) \rangle \).

Sometimes I will abuse language by talking of elements and cardinality of related sets \( \langle X, R \rangle \), \( \langle X', R' \rangle \) are isomorphic (to each other), if there is a one-to-one map \( \phi \) from \( X \) onto \( X' \) such that for any \( x, y \in X \)

\[ x R y \quad \text{if and only if} \quad \phi(x) R' \phi(y). \]

A least element in a related set \( \langle X, R \rangle \) is an \( x \in X \) such that \( x R y \) for any other \( y \in X \). If instead \( y R x \), then \( x \) is a greatest element.

By a (strict) linear order on some set \( Y \) I understand a transitive binary relation \( R \) on \( Y \) such that for any \( x, y \in Y \) exactly one of \( x R y, y R x \) and \( x = y \) holds (this mixes irreflexivity, connectedness, and even, redundantly, asymmetry). In this case, I will call any related set \( \langle X, R \rangle \) such that \( X \subseteq Y \) a linearly ordered set, or, for short, a loreset.

\( \langle \mathbb{R}, < \rangle \) will denote the set of all real numbers linearly ordered by the binary relation of being less. By a real interval I will understand a set \( I \subseteq \mathbb{R} \) such that

\[ \text{if } s, t \in I \text{ and } s < r < t \text{ then } r \in I. \]

I am going to call a real interval open if it contains neither a least nor a greatest element, half-open if it contains a least or a greatest element but not

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9 Formulas like these and others are just meant to be shorthand for English mathematical expressions; they are not meant to refer to expressions of a formal language from the realm of metamathematics. However, metamathematical questions may rise eventually, and one will then be able easily to realize that some condition discussed can be formalized in some first-order language, e.g.

\( R \) is a binary relation on \( X \), if it is one between \( X \) and \( X \).

10 Now that [40] is a relevant reference, it should be noted that transitivity here—as will be usual for many readers—means that \( x R z \) whenever \( x R y \) and \( y R z \); it should not be confused with what in [40] and other contexts is denoted by the same term. In these contexts, ‘transitive’ could have been replaced by ‘homogeneous’.—By the trichotomy condition, the union of domain and codomain of \( R \) is the only set on which a given binary relation \( R \) can be a linear order.

11 While ‘poset’ is a standard term in the literature on partially ordered sets, I have never seen this term ‘loset’. For didactical reasons, however, I urgently need a short term.

‘Chain’ is short as well, but should work like ‘antichain’ and therefore does not fit.
both, and compact if it contains both a least and a greatest element (which may coincide). By a non-trivial real interval I understand a real interval having more than one element—which is the same as having the cardinality of \( \mathbb{R} \). Observe that being non-trivial is the same as being non-void in case of an open real interval and as having distinct borders in the case of a compact real interval, while all half-open intervals are non-trivial.

3 Some events, some Russellian instants.

For the remaining, I fix some non-void set \( E \) of so-called “events” and a binary relation \( P \subseteq E \times E \) on it, the philosophical meaning of which is to be ‘wholly precedes’. Proofs of what is claimed on the few lines following are indicated in the first two subsections of Section 8.

An antichain (in \( \langle E, P \rangle \)) is a subset \( A \) of \( E \) such that for any two \( a, b \in A \) neither \( a \ P \ b \) nor \( b \ P \ a \). A maximal antichain is an antichain that cannot be extended to another antichain by adding any non-void set of events. By the axiom of choice, any antichain can be extended to a maximal antichain, and as the empty set is an antichain, there is at least some maximal antichain. For the same time that \( \langle E, P \rangle \) is fixed, let \( L \) be just the set of maximal antichains in \( \langle E, P \rangle \). By Russell’s proposal, the elements of \( L \) are instants of time as “constructed” from those “events” presented in \( E \) and from their temporal relationships encoded by \( P \). (Therefore, it will be informal variables like \( s, t \) that range in \( L \), while \( A, B, \ldots \) will range over arbitrary subsets of \( E \).)

The “order of time”, then, is defined to be

\[
T := \{ \langle s, t \rangle \in L \times L \mid a \ P \ b \text{ for some } a \in s \text{ and some } b \in t \}.
\]

In “order notation” and “philosophical diction”, \( s \ T \ t \) (only) for “instants” \( s, t \) in \( L \) just if some event \( a \) in \( s \) “wholly precedes” some event \( b \) in \( t \).

I furthermore fix that \( P \) be irreflexive (i.e., no event precede itself) and that for any events \( a, b, c, d \),

\[
\text{if } a \ P \ b \text{ and } c \ P \ d \text{ then } a \ P \ d \text{ or } c \ P \ b. \tag{I}
\]

\( \langle E, P \rangle \) is now an interval order in the sense of [16] and [17]; and I may use results from Wiener [49], who calls a \( P \) satisfying an equivalent pair of conditions a ‘relation of complete sequence’, as well as from Russell’s [43], who uses a triple of conditions equivalent to each of the previously mentioned pairs.) Reading (I) as ‘if \( a \) wholly precedes \( b \) and \( c \) wholly precedes \( d \), but \( c \) does not wholly precede \( b \), then a wholly precede \( b \)’ may exhibit (I) to

\[12\]For distinguishing Russell’s from A. N. Whitehead’s credits concerning constructing “points” and “instants”, cf. [1, pp. 252f.].

\[13\]Wiener’s paper is summarized in modern notation by [18].
be a ‘self-evident’ feature of $P$ if the latter is read as ‘wholly precedes’.\textsuperscript{14} This would render $\langle E, P \rangle$ quite a good starting point for philosophically justifying instants of time. In fact, in the situation of the hypothesis of the proposed reading, (i) either $b$ wholly precedes $c$, and then by “self-evident” transitivity of ‘wholly precedes’ $a$ wholly precedes $d$; (ii) or $b$ and $c$ have “something in common” which must be preceded by $a$ and precede $d$, and this might be considered some “evidence” for a wholly preceding $d$.\textsuperscript{15}

By (I) and irreflexivity of $P$, the latter relation is transitive. Moreover, (I) ensures that $T$ is a linear order on $L$ [49, p. 445f.]. In this respect, $T$ is an adequate mathematical reconstruction of temporal precedence between instants of time. Some aspects of adequacy, however, remain to be considered.

4 Core Task.

I am going to propose a few characterization problems to be solved, indicating them as ‘questions’.

First compare the characterization problem that Thomason [46] deals with. He explains how Walker would construct, instead of $\langle L, T \rangle$, another time ordering $\langle L', T' \rangle$ from $\langle E, P \rangle$. He then presents a necessary and sufficient condition on $\langle E, P \rangle$ for $\langle L', T' \rangle$ being isomorphic to $\langle R, < \rangle$. The analogous problem for $\langle L, T \rangle$ he considers ‘difficult’, viz.,

**Question 1.** When, in terms of $\langle E, P \rangle$, is $\langle L, T \rangle$ isomorphic to $\langle R, < \rangle$?

Here, I am using ‘when . . . ’ to abridge something like ‘What conditions on $\langle E, P \rangle$ are necessary and sufficient for $\langle E, P \rangle$ being isomorphic to $\langle R, < \rangle$?’, and similarly below. These questions are, admittedly, not perfectly precise, but I am not attempting to explicate the notion of ‘characterization’ in this paper; it will suffice to present convincing solutions to the problems, however “ill-posed” the latter may be. We will encounter solutions for other characterization problems by conditions which could be formalized as first-order sentences in a language interpretable in $\langle E, P \rangle$. It may be no surprise in connection with “the continuum”, however, that the main characterizing conditions are not first-order, but stipulate a countable set of objects from $\langle E, P \rangle$ having a certain property (cf., e.g., [35, Theorem 2.1]).

**Variants** of Question 1 may be interesting from a physical, mathematical, or perhaps even philosophical point of view. First, one might really want an explanation of “time as a continuum”, but one might disagree with

\textsuperscript{14} This is, in only a slightly different situation, held by [1, p. 252]; also cf. [30, p. 181].
\textsuperscript{15} Cf. [30, p. 181]. For further interpretations or applications of interval orders, cf. [17, pp. 20ff.].—In philosophically justifying instants of time, however, it is difficult to say what it is that $b$ and $c$ “have in common” in the second case without committing a petitio principii. I am afraid the only convincing motivations for (I) refer to how interval structures are represented by “point structures” like the present $\langle L, T \rangle$. 
Thomason as to what a *continuum* is. Indeed, the purportedly well-known order-type of “*the* continuum” that Cantor characterized in his celebrated [11] was actually that of the *closed* (i.e., compact) unit interval, and another notion of “continua”, entailing topological compactness as well, was prominent in a branch of early topology, so-called curve theory (cf. Subsection 9.2). So (as some readers conclude immediately and others will see soon) a continuum may be—and has been—considered something essentially different to a structure comprising all the real numbers. Unlike the real numbers, it might have a first point, a last point, or both.

On the other hand, the question of what *a* or *the* continuum is might be considered too “theoretical”, namely “only” of mathematical or even philosophical interest. It might be considered more important whether the construction renders time as it is used in *physics*. Indeed, physical textbooks seem to postulate that instants of time are just the same as real numbers. However, from the viewpoint of physical, relativistic cosmology (of General Relativity, i.e.) which prevails nowadays, astronomical observations indicate that not *every* real number corresponds to an instant of time (in the usual way): a ‘Big Bang’ is believed to have *started* time; and a final collapse of the universe is reckoned with to *end* time.\(^\text{16}\)

However, this reasoning may only lead to consider bounded open real intervals instead of all the real numbers, and this switch does not change anything at least order-theoretically or topologically. First or last instants which would make a difference seem to play no role in physics—you usually encounter open time intervals in textbooks, even the Big Bang has no starting instant of time.\(^\text{17}\) Yet, *some* physicist might one day leave the herd.

So there may be some, if only little, reason to consider more general versions of Question 1. The additional answers will not need extra mathematical effort. From the results the reader may choose what he likes most. The

\(^{16}\) This, however, stems from the special role time is given in the mathematical rendering of how the universe seems to behave, and talking of “time” perhaps only makes sense when spacetime is a “product” of time and space in some special sense (I do not expect spacetime singularities of black holes that are formed eventually or that dissolve by a reasoning of Stephen Hawking [21] would deprive the picture of *one* time line for all “point” events of spacetime of use). Of course, in some “metaphysical” view “Time” could have existed earlier than the Big Bang and could since then somehow “interplay” with “mathematical-physical time”. Such a “metaphysical everexisting Time” would be of no empirical value, only, maybe, of some aesthetical one—if someone’s taste behaves appropriately. What is most important here: such a “metaphysical Time outside of physical time” could not be reconstructed from “empirical” events! “Empirical” events occur after the Big Bang and before a final collapse and will therefore only detect the instants of—bounded—*physical* time. Indeed, Walker’s approach presented by [46] was part of justifying, by assumptions about simple observations, a mathematical model of the whole development of the universe according to General Relativity (cf. references in [46]). The spacetimes according to this mathematical model are nowadays called ‘Robertson–Walker spacetimes’ (cf. [37, pp. 341ff.], e.g.).

\(^{17}\) Cf. footnote 16.
reader may like to know something about “continua”, and she may conceive of continua so that two of them are isomorphic to each other (like the open real intervals), or so that there are two or more isomorphism types of continua. E.g., a continuum might be viewed to be anything order-isomorphic to some non-trivial real interval. Consider

**Question-Scheme 1**. When, in terms of \( \langle E, P \rangle \), is \( \langle L, T \rangle \) isomorphic to \( \langle I, \langle \rangle \rangle \)?

This “scheme” produces one question for each real interval \( I \). Question 1 is that example where \( I = \mathbb{R} \). The “scheme” represents readers with somewhat very “narrow” conceptions of continuum likeness. As for an example of a broad conception, its answers enable answering

**Question 1**. When, in terms of \( \langle E, P \rangle \), is \( \langle L, T \rangle \) isomorphic to some \( \langle I, \langle \rangle \rangle \) where \( I \) is a non-trivial real interval?

## 5 Via completeness and dense subsets.

The following is well known:\textsuperscript{18}

**Fact 1.** A related set is isomorphic to \( \langle \mathbb{R}, \langle \rangle \rangle \) if and only if it is a non-void, complete and separable loset having neither a least nor a greatest element.

To understand this, recall some definitions. Let \( \langle X, R \rangle \) be some related set again.

A *lower bound* of some \( Y \subseteq X \) is an \( x \in X \) which is a least element of \( \{x\} \cup Y \) (\( x \) may be element of \( Y \)). A *greatest lower bound* of \( Y \subseteq X \) is a greatest element of the set of all lower bounds of \( Y \). By analogy, an *upper bound* of \( Y \) is greatest element of the union of its singleton with \( Y \), and a *least upper bound* is a least element of the set of all upper bounds. (If \( \langle X, R \rangle \) is a loset, \( Y \) has at most one greatest lower or, resp., least upper bound, of course.)

Now, \( \langle X, R \rangle \) is *complete* if every non-void subset of \( X \) having a lower bound has a greatest such and every subset of \( X \) having an upper bound has a least such.\textsuperscript{19}

A set \( Y \) is *dense* in \( \langle X, R \rangle \) if \( Y \subseteq X \) and for any two \( x, z \in X \) such that \( x R z \) there is some \( y \in Y \) such that \( x R y R z \).

\textsuperscript{18}According to [46, p. 94] (where denseness is redundant, viz., following from existence of a dense subset); cf. [40, pp. 33–37], where instead of Exercise 2.29 it suffices to think of the set of the rational numbers as a countable set dense in \( \langle \mathbb{R}, \langle \rangle \rangle \).

\textsuperscript{19}For the sake of consistency, please forget about what a graph theorist understands by a complete graph. Consistency is only endangered at first sight as long as—like in [17, p. 1]—‘complete’ is used meaning ‘connected’ when talking of binary relations. But, again, beware of calling a “completely ordered set”, e.g., ‘complete’.
Finally, \((X, R)\) is separable if there is a countable\(^{20}\) subset of \(X\) being dense in \((X, R)\).\(^{21}\)

In view of Fact 1, and since \((L, T)\) is a non-void loset anyway (by Section 3), we approach our goal by moving from Question 1 to

**Question 1’.** When, in terms of \((E, P)\), is \((L, T)\) complete and separable without least or greatest element?

Question 1’ can be split into the following questions, which will be dealt with separately from each other.

**Question 2.** When, in terms of \((E, P)\), is \((L, T)\) complete?

**Question 3.** When, in terms of \((E, P)\), is \((L, T)\) separable?

**Question 4a.** When, in terms of \((E, P)\), has \((L, T)\) a least element?

**Question 4b.** When, in terms of \((E, P)\), has \((L, T)\) a greatest element?

To deal with Question Scheme 1*, Fact 1 can be generalized to arbitrary real intervals—taking some subtleties (as observed in Section 2) into account:

**Fact 1*.** Let \(I\) be a real interval.

(a) Assume \(I\) is open. If \(I\) is empty, then \((I, <) = (\emptyset, \emptyset)\), and this is the only related set isomorphic to \((I, <)\). Otherwise a related set is isomorphic to \((I, <)\) if and only if it is a non-void complete and separable loset having neither a least nor a greatest element.

(b) If \(I\) contains a least element, but no greatest, [the other way round, resp.], a related set is isomorphic to \((I, <)\) if and only if it is a complete and separable loset with\(^{22}\) a least [greatest, resp.] and without greatest [least, resp.] element.

(c) Assume \(I\) is compact.\(^ {23}\) If \(I\) contains just one element, then a related set is isomorphic to \((I, <)\) if and only if it is \((\{x\}, \emptyset)\) for some object \(x\). Otherwise a related set is isomorphic to \((I, <)\) if and only if it is a complete and separable loset with a least and a greatest element where the latter do not coincide.\(^ {24}\)

\(^{20}\)By my understanding here, ‘countable’ does not imply ‘infinite’. So some questions of cardinality can be discussed separately below.

\(^{21}\)Cf. [40, Definition 2.28].

\(^{22}\)So ‘non-void’ is not needed.

\(^{23}\)My definition slightly differed from the usual topological one, and \(I \neq \emptyset\) is implied here.

\(^{24}\)According to [40, p. 40], the non-trivial cases may be proved essentially the same way as Fact 1. Alternatively, one could derive them from Fact 1 by observing what is preserved when one removes or adds least or greatest elements. The enumeration of order types in [40, p. 40] is imprecise, if not incomplete.
So there are four types (with respect to order-isomorphisms) of non-trivial real intervals: open, half-open (two types), and compact-with-more-than-one-element. Questions 2, 3, 4a, and 4b suffice to recognize the first three cases for \(\langle L, T \rangle\) (corresponding to cases (a) and (b) of Fact 1*) from looking at \(\langle E, P \rangle\). For recognizing the fourth case, it suffices to deal with the following final question.

**Question 5.** When, in terms of \(\langle E, P \rangle\), has \(L\) more than one element?

An obvious “answer-scheme” to Question-Scheme 1* arises, following the lines of Fact 1*. An answer to Question 1+ derives which may be considered a consequence of the following fact, which is entailed by Fact 1*.

**Fact 1+.** A related set is isomorphic to \(\langle I, < \rangle\) for some non-trivial real interval \(I\) if and only if it is a complete and separable loset having more than one element.

### 6 Pivotal derived notions.

I am going to introduce further relations on the set \(E\) of events in order to state some conditions more succinctly than I could do without them. At the same time, some visualizing possible situations may be in order so that it is easier to understand what I mean.

Assume for a moment \(E\) is a three-element set \(\{a, b, c\}\) and \(P = \{(a, b)\}\). This situation is visualized by Figure 1.25 a \(P\) b is visualized by arranging

\[
\begin{array}{c}
\quad a \\
\downarrow \\
\quad c \\
\quad b \\
\end{array}
\]

“direction of time”

Figure 1: Three events.

horizontal strokes representing \(a\) and \(b\) so that a *vertical* stroke can be filled in right-hand to the horizontal stroke representing \(a\) and left-hand to the horizontal stroke which represents \(b\).

By contrast, no vertical stroke would have two horizontal strokes on different sides such that one of them would represent \(a\) and the other would represent \(c\). This holds for \(b\) in place of \(a\), as well. Rather, \(c\) is “overlapping” \(a\) as well as \(b\).

(By the way, \(\{a, c\}\) and \(\{b, c\}\) are antichains; maximal antichains, in fact, and the only ones. So they form the set \(L\) of instants of time, and we have \(\{a, c\} T \{b, c\}\).)

25Cf. [46, p. 88].
Furthermore, c “begins earlier than” b, “witnessed” by a overlapping c but wholly preceding b. Similarly, a “ends earlier than” c, “witnessed” by b overlapping c but being wholly preceded by a.

The above ‘moment’ (specializing \( \langle E, P \rangle \)) is over, and I generalize the situation by further definitions of binary relations on E representing the relationships observed above.

\[
\begin{align*}
S & := \{ \langle a, b \rangle \in E \times E \mid \text{neither } a P b \text{ nor } b P a \}; \\
SP & := \{ \langle c, b \rangle \in E \times E \mid c S a P b \text{ for some } a \}; \quad (26) \\
PS & := \{ \langle a, c \rangle \in E \times E \mid a P b S c \text{ for some } b \}. \quad (27)
\end{align*}
\]

(Visualize by help of Figure 1.) Now \( aSb \) is to mean that \( a \) overlaps \( b \), the other way round, or just that \( a \) and \( b \) overlap, however you like:\( c SP b \) is to mean that \( c \) begins earlier than \( b \); and \( a PS c \) is to mean that \( a \) ends earlier than \( c \).

An event \( a \) will be called an SP-minimal element of a subset \( A \) of \( E \), if \( a \in A \) and there is no \( b \in A \) such that \( b SP a \). “Dually”, an event \( a \) will be called a PS-maximal element of \( A \), if \( a \in A \) and there is no \( b \in A \) such that \( aPS b \).

The relations on \( E \) defined just before make it easier to define the notion of ‘having a first [last, resp.] instant’ in one line, which in [43] plays in important role for the question of the existence of instants. This notion is vital for formulating answers to all my questions but one as well. To enable the reader to make sense of my definitions following, I precede an outline of Russell’s [43] discovery in terms of his philosophical interpretation.

Call an event \( b \) a contemporary of some event \( a \) whenever they overlap (purely mathematically: \( a S b \)). Call \( b \) an initial contemporary of \( a \) if additionally \( a \) does not begin earlier than \( b \) (not \( aSP b \)). Now Russell found out if such an \( a \) exists, it is an element of \( E \) by the definition of a related set or, sufficing as well, by the definition of \( S \).

Of course, \( SP \) and \( PS \) are just the compositions of the binary relations \( P \) and \( S \) and vice versa, resp., in the sense of [17, p. 3].

Note: a \( S b \) if and only if \{a, b\} is an antichain; and a subset \( A \) of \( E \) is an antichain if and only if a \( S b \) for all \( a, b \in A \).

I am nothing but reporting some notation and terminology introduced by Russell in [43].—\( \langle E, P, SP, PS \rangle \) is now an ‘event ordering’ in the sense of Thomason [46, Definition 1].

The notion of ‘duality’ can be made rather precise, cf. [2, p. 13]. I could have made it quite precise if I had introduced a formal language. Without, just think of the dual of a “statement” (or a “condition”) as the result of interchanging the event symbols on both sides of \( P \), if \( P \) is the only relation symbol (besides \( = \)) occurring and if conjunctions are “written out” instead of using “chain notation”. Using “chain notation” for conjunctions, consider a one-term conjunction a “conjunction chain” as well; then the dual is the result of reversing all “conjunction chains”—still if no symbols for derived notions occur. Otherwise, \( S \) remains unchanged, while \( SP \) is replaced by \( PS \) and vice versa, and ’SP-minimal’ is replaced by ’PS-maximal’ and vice versa. Finally the symbol \( \exists - \) introduced soon has to be replaced by \( \exists + \) and vice versa.

SP-minimal and PS-maximal elements may exist because, by Lemma 1 below, \( SP \) and (dually) \( PS \) are irreflexive.
that an event \( e \) has a first instant (i.e., there is a \( T \)-least maximal antichain containing \( e \)) if and only if (keep the following in mind for a moment), whenever \( e \) begins earlier than some event \( a \), this \( a \) is wholly preceded by some initial contemporary of \( e \). (In this case, the set of initial contemporaries of \( e \) is that first instant.)

To make mathematical use of the notion of ‘having a first instant’, I introduce a symbol denoting its extension (as far as \( E \) is concerned):

\[
\exists_\rightarrow := \{ e \in E \mid \text{whenever } e \mathrel{SP} a, e \mathrel{(S \setminus SP)} b \mathrel{P} a \text{ for some } b \}.
\]

So \( e \in \exists_\rightarrow \) “philosophically” means that \( e \) has a first instant.\(^{32}\)

Dually,

\[
\exists_\leftarrow := \{ e \in E \mid \text{whenever } a \mathrel{PS} e, a \mathrel{P} b \mathrel{(S \setminus PS)} e \text{ for some } b \}
\]

contains all events having a last instant.

(However, I am not quite sure about whether it was really \textit{Russell} who found that \( e \in \exists_\rightarrow \) if and only if \( e \) has a first instant, and about what actually \textit{Wiener} contributed to the result.)\(^{33}\)

\(^{32}\)I prefer writing \( e \in \exists_\rightarrow \) (and introducing that notation) to just writing ‘\( e \) has a first instant’ in order to call to the reader’s mind that the notion I am making use of is a purely mathematical one. Moreover, to make clear that I am answering questions “in terms of” \( \langle E, P \rangle \) in the sense of Section 4), I want to emphasize that I am merely talking of elements \( e \) of \( E \) such that, \( e \mathrel{(S \setminus SP)} b \mathrel{P} a \text{ for some } b \) whenever \( e \mathrel{SP} a \); I am not begging the question by talking about maximal antichains containing \( e \).

\(^{33}\)Russell once claimed the result was his own, but his references indicate the possibility of him only having conjectured it, while it was \textit{Wiener} who found the proof. To be precise:

(i) At the beginning of [43], Russell credits Wiener [49] with having shown what conditions are necessary in order that maximal antichains should form a linearly ordered set as outlined above. By his next sentence, Russell claims \textit{he} (himself) had shown that every event has a first instant ‘if’ it satisfies the condition stated above and defining \( \exists_\rightarrow \)—referring to his [42]. A few pages behind, he proves the converse.

(ii) Russell’s [42, Lecture IV], however, where the subject is discussed, offers no mathematical proof of anything. Instead, one paragraph only \textit{claims} ‘[i]t will be found’ that the condition ensures an event occurring at the set of its initial contemporaries as its first instant. ‘For a mathematico-logical treatment’, a footnote refers to Wiener’s [49]!

(iii) In [49, p. 447f.], Wiener proves that, if an event satisfies the condition, then the set of its initial contemporaries is a maximal antichain containing that event (i.e., what Russell claimed to have shown). In advance, Wiener ascribes to Russell just having \textit{formulated} the condition.

What both Russell and Wiener tell concerning priority could perhaps be reconciled by another possibility, viz., that Russell \textit{did} prove the equivalence of his condition on an event to the latter having a first instant earlier than Wiener, but without publishing his formal work at that time; while Wiener’s [49] proof followed quite different lines. This guess rests on the footnote of [49, p. 441], where Wiener tells that the paper (as a whole) resulted from attempting to simplify and generalize what Russell had done so far. The fact that Russell in [43] referred wrongly to his [42] then might be some “inaccuracy” or “slip of memory”, which I, however, hardly can conceive in the light of the editor’s (R. C. Marsh’s) Preface to the collection to which [43] belongs: ‘Lord Russell has been consulted on all
7 Solutions.

Non-constructive\(^{34}\) assumptions used. As a whole, the ensuing answers to the questions asked in sections 4 and 5 assume the axiom of choice or, at least, the principle of countable choice\(^{35}\) and that every antichain (in our \(\langle E, P \rangle \)) of at most 2 elements extends to a maximal antichain. Theorem 1, however, needs the latter assumption for singletons in place of antichains only, and non-voidness of \(L\) (Section 3) merely means existence of a maximal antichain with no regard to what elements of \(E\) it should contain.—I call \(\langle E, P \rangle\) \(n\)-maximizing if every antichain of at most \(n\) elements extends to a maximal antichain; the assumptions mentioned before refer to this for \(n = 2, 1, 0\), respectively.\(^{36}\)—Moreover, the condition of Theorem 3 characterizing separability of \(\langle L, T \rangle\) implies that \(\langle E, P \rangle\) is 2-maximizing and thus (whenever assumed) removes any reliance on a non-constructive principle as far as existence of maximal antichains is concerned; in particular, no maximizingness assumption is involved in sufficiency of the characterization of real intervals.—Now, I believe that ‘\(\langle E, P \rangle\) is 2-maximizing’ or any of some other consequences of the axiom of choice discussed in Subsection 8.2 below does not imply the principle of countable choice. In case this belief is correct, it is important to note that the latter principle is used for both directions of Theorem 3. Under the previous separability assumption on \(\langle E, P \rangle\) it remains the only non-constructive principle underlying one direction of Theorem 3 and Corollary 2—I name these assumptions and the directions in which they are used in parentheses near each statement below.

Question 5 is most easily answered, so start with

**Theorem 1.** \(L\) has more than one element if and only if \(P \neq \emptyset\).

(This uses just that \(\langle E, P \rangle\) is 1-maximizing for ‘if’.)

Answers to Questions 4a and 4b are slightly more difficult:

**Theorem 2.** (a) \(\langle L, T \rangle\) has a least element if and only if \(E\) has an \(SP\)-minimal element contained in \(\exists_\cdot\) (‘having a first instant’, i.e.).

(b) (dually to the previous) \(\langle L, T \rangle\) has a greatest element if and only if \(E\) has a \(PS\)-maximal element contained in \(\exists_+\) (‘having a last instant’, i.e.).

matters relating to the text of the papers, and these, to the best of my knowledge, are here issued in the form which he wishes to be taken as final and definitive.’

It might be noteworthy in this context that [43], before having appeared in 1956 in the collection I am referring to, originally appeared in 1936 in the Cambridge Proceedings; while its references [42] (1st ed.) and [49] had appeared in 1914.

For distinguishing Russell’s from Wiener’s credits in general, cf. footnote 7 above.

\(^{34}\)What ‘constructive’ and ‘non-constructive’ mean here and below is rendered more precise in Subsection 8.1. For a first approximation, “constructivity” avoids the axiom of choice and any weak variant of it.

\(^{35}\)Cf. [29, pp. 167, 2.1]. However, this principle even has been accepted in (parts of) constructive mathematics [45].

\(^{36}\)Choose an \(SP\)-maximal and a \(PS\)-minimal element from an antichain to see that \(\langle E, P \rangle\) is \(n\)-maximizing for every non-negative integer \(n\) as soon as it is 2-maximizing.
(Both (a) and (b) use that \( \langle E, P \rangle \) is 2-maximizing for ‘only if’.)

In the following, situations like \( a \ P c \ S d \ P b \) are mentioned repeatedly. Such a situation is illustrated in Figure 2. (\( a \ P b \) follows by (I).) In such

\[
\begin{array}{cccc}
  a & & & b \\
    & c & & \\
    & & d \\
\end{array}
\]

“direction of time”

Figure 2: Antichain \( \{c, d\} \) “in gap” of chain \( \{a, b\} \).

a situation, \( \{c, d\} \) is an antichain “in the gap” of the chain \( \{a, b\} \) (where a chain is a subset of \( E \) forming a loset when \( P \) is restricted to it). If \( E \) were just \( \{a, b, c, d\} \), \( L \) would be \( \{\{a, d\}, \{c, d\}, \{b, c\}\} \), and \( T \) would do \( \{a, d\} \ T \{c, d\} \ T \{b, c\} \), so \( \{c, d\} \) would be an “instant” \( T \)-between “all” the instants containing \( a \) in one direction and “all” the instants containing \( b \) in the other.—This enables answering Question 3 as follows.

**Theorem 3.** \( \langle L, T \rangle \) is separable if and only if there is a countable subset \( D \) of \( E \) such that for all \( \langle a, b \rangle \in P \) “at the same time”

(i) if \( a \in \exists_+ \) and \( b \in \exists_- \), then there are \( c, d \in D \) such that \( a \ P c \ S d \ P b \);

(ii) if \( a \notin \exists_+ \) then, whenever \( e \ PS a \), there are \( c, d \in D \) such that \( e \ P c \ S d \ P b \);

(iii) (dually to the previous condition) if \( b \notin \exists_- \) then, whenever \( b \ SP e \), there are \( c, d \in D \) such that \( a \ P d \ S c \ P e \).

(This uses countable choice in both directions and ‘\( \langle E, P \rangle \) is 2-maximizing’ for ‘only if’. In proving the theorem, I will generalize it to infinite cardinalities of dense subsets, then using the full axiom of choice.)

Here is one by-product announced in the beginning:

**Corollary 1.** \( \langle L, T \rangle \) is dense (i.e., \( L \) is dense in \( \langle L, T \rangle \)) if and only if for all \( \langle a, b \rangle \in P \) such that \( a \in \exists_+ \) and \( b \in \exists_- \) there are \( c, d \in E \) such that \( a \ P c \ S d \ P b \) (compare first condition in the above theorem).

(This uses that \( \langle E, P \rangle \) is 2-maximizing for ‘only if’.)* The criterion “in terms of” \( \langle E, P \rangle \) for \( \langle L, T \rangle \) being dense given here is a weakened version of a condition stated by Russell in a footnote of [42, Lecture IV] and by Wiener in [49, pp. 446f.],\(^\text{37}\) viz.,

\[
\text{for all } \langle a, b \rangle \in P, \text{ there are } c, d \in E \text{ such that } a \ P c \ S d \ P b. \quad \text{(III)}
\]

\(^{37}\)Both called—according to the terminology of *Principia Mathematica* [48, *p270*]—‘compact’ what nowadays is called ‘dense’ (or ‘dense-in-itself’); only Anderson in [1, pp. 256f.] still sticks to that historical term which was attaining a very different meaning in mathematics (topology) at the same time—going back to a 1906 thesis by M. Fréchet, cf. Hausdorff 1914 [20] and [13, Chapter V]. According to this meaning, compactness of time would imply that there is a very first and a very last instant.
This, of course, is only a sufficient, no necessary condition for \( \langle L, T \rangle \) being dense.\(^{38}\) Wiener in [49, pp. 447ff.], however, uses an additional premise to prove that \( \langle L, T \rangle \) is dense, viz., \( \exists_\_ = E \) (in my terms, meaning ‘every event has a first instant’ as indicated in Section 6 above). Thus he does not consider (III) sufficient as I do. But Wiener uses \( \exists_\_ = E \) only to prove that the postulated two-element antichain “in the gap” of a two-element chain extends to a maximal antichain. By contrast, the present paper assumes maximal-antichain extendibility of any two-element antichain \((E, P)\) 2-maximizing) from the start, so the crucial step in deriving denseness of \( \langle L, T \rangle \) from (III) as mentioned goes without invoking Wiener’s (in fact, Russell’s)\(^{39}\) additional premise \( \exists_\_ = E \).—In [43], Russell presents a condition on \( \langle E, P \rangle \) necessary for \( \langle L, T \rangle \) being dense and a pair of conditions he claims to be sufficient. A simple counter-example presented by Anderson [1, p. 256ff.] shows that this claim is wrong.\(^{40}\) As a remedy, Anderson suggests

\(^{38}\)To see that (III) is no necessary condition for \( \langle L, T \rangle \) being dense, let for every pair \( z_1, z_2 \) of integers \( I(z_1, z_2) \) be the real interval \( \{ r \in \mathbb{R} \mid z_1 2^{z_2} - 2^{z_2 - 1} < r \leq z_1 2^{z_2} \} \). (Note that \( \bigcup \{ I(z_1, z_2) \mid z_2 \text{ integer} \} \) is only “half” of \( \mathbb{R} \) for each \( z_2 \).) Then let \( E \) be the set of all these intervals and let \( P \) be the relation on \( E \) naturally deriving from \( < \). Now, e.g., \( I(1,0), I(1,1) \) are \( P \)-neighbours; but \( \langle L, T \rangle \) turns out to be isomorphic to \( \langle \mathbb{R}, < \rangle \) (map each real number \( r \) to the set of “events” containing \( r \)) and so is dense, since \( \langle \mathbb{R}, < \rangle \) is dense. (No “event” as chosen like this has a first instant, since there are no initial contemporaries ending earlier, but lots of contemporaries beginning later. Replacing \( \leq \) in the definition of \( I(z_1, z_2) \) by \( < \) would make no difference but the need to recognize that, now, each \( I(z_1, z_2) \) is element of that maximal antichain which contains those “events” that include \( z_1 2^{z_2} \).

Continuing the example (forget about the variation just mentioned) shows that \( \langle E, P \rangle \) needs not to be K-dense (cf. [6, p. 76]), which following [19] also has been called ‘CAC’ (chain-antichain-complete), in order that \( \langle L, T \rangle \) be as wanted: \( \{ I(2^n - 1, -n) \mid n \text{ positive integer} \} \) is a chain. By the axiom of choice—but also in a constructive way—it can be extended to a maximal chain. But no such extension intersects with the maximal antichain of “events” including 1. (One constructive way of extending the chain to a maximal antichain would be adding all \( I(2^n - 1, -n + m) \) for further positive integers \( m \); if \( C \) then is the chain obtained so far, \( \{ \{ r + z \mid r \in M \} \mid M \in C, z \text{ integer} \} \) is a maximal chain. It intersects with no maximal antichain that corresponds to an integer.)

\(^{39}\)Wiener [49, p. 447] attributes the condition to Russell.

\(^{40}\)One of the conditions Russell [43] claims to be sufficient is just that \( \langle E, P \rangle \) is 2-maximizing—the surrogate for the axiom of choice which the present paper uses as well. The other condition is that no event lasts only for an instant. This was Russell's epistemological starting point already in [42, Lecture IV] (cf. his later [44, p. 293]). It implies the first-order condition \( \Delta \subseteq SPS \) where \( \Delta = \{ \langle a, b \rangle \in E \times E \mid a = b \} \) and \( SPS \) is the composition of relations according to the well-known notation (cf. [17, p. 3]) like \( SP \) and \( PS \) and—below in Subsection 8.3—\( PSP \). The converse implication holds when \( \langle E, P \rangle \) is 2-maximizing (Russell's first condition). \( \Delta \subseteq SPS \) has just nothing to do with linear denseness of \( \langle L, T \rangle \)—i.e., these two conditions are independent (for the direction not dealt with by Anderson, take the rational numbers for \( E \) and \( T = < \cap (E \times E) \); then \( \langle L, T \rangle = \langle E, P \rangle \) is dense, though not \( \Delta \subseteq SPS \)—here \( S = \Delta \).) It should not be “supplemented” as in Kleinknecht [26], but just ignored. (Even less one should add an assumption like [26, A1.77] which contradicts Russell’s no event lasts only for an instant—Kleinknecht assumes that the members of each antichain \( t \) have a common part \( c \); but if \( t \) is a maximal antichain, this event \( c \) would last for its only instant \( t \), since all \( d, d' \) overlap-
for all \( \langle a, b \rangle \in P \), there is \( c \in E \) such that \( a \; P \; c \; P \; b \)
as an alternative to the conjunction of (III) and \( \exists_\rightarrow E \) for entailing that
\( \langle L, T \rangle \) is dense—obviously not realizing that under his [1, p. 255] assumption
of the axiom of choice (which entails \( \langle E, P \rangle \) 2-maximizing, see Subsection 8.2
below) (III) alone is a condition entailing that \( \langle L, T \rangle \) is dense and a (much)
weaker one than his own proposal.—Thus, to my knowledge, the statement
of the previous corollary is new.\(^{41}\)

To tackle **Question 2**, define for every \( A \subseteq E \)

\[
\begin{align*}
U(A) & := \{ e \in E \mid a \; P \; e \text{ for all } a \in A \}; \\
H(A) & := \{ e \in E \mid e \; P \; b \text{ for all } b \in U(A) \}.
\end{align*}
\]

\(U(A)\) comprises all those events that are wholly preceded by all events of
\(A\). \(H(A)\), by contrast, is some kind of “hull” of \(A\) collecting not only all
those events that do not end later than some event from \(A\) (so, of course,
\(A\) is a subset of \(H(A)\)), but collects even those events that do end later
than all events from \(A\), but only “non-uniformly”.\(^{42}\) \(\langle H(A), U(A) \rangle\) is a pair
of maximal sets of events such that all the events of the first set wholly
precede all the events of the second set. Every pair with this property is

---

\(^{41}\)I.e., \(b \in H(A)\) if \(a \; P \; c \; S \; b\) for all \(a \in A\) and some \(c\) “depending” on \(a\), without
there being a “single” \(c\) such that \(a \; P \; c \; S \; b\) for all \(a \in A\).—It is also easy to see that the
operator \(H\) is idempotent and monotone.
\( \langle H(A), U(A) \rangle \) for some \( A \subseteq E \), and the property generalizes the notion of a Dedekind cut (in the sense of [40, Definition 2.22], e.g.).

My answer to Question 2 then is

**Theorem 4.** \( \langle L, T \rangle \) is complete if and only if every \( A \subseteq E \) satisfies one of the following conditions:

(i) \( A \) or \( U(A) \) is empty;

(ii) \( H(A) \) has a \( PS \)-maximal element contained in \( \exists_+ \);

(iii) \( U(A) \) has an \( SP \)-minimal element contained in \( \exists_- \);

(iv) for every \( a \in H(A) \) there is some \( c \) neither in \( U(A) \) nor in \( H(A) \) such that \( a \ P \ c \), and for every \( b \in U(A) \) there is some \( d \) neither in \( U(A) \) nor in \( H(A) \) such that \( d \ P b \).

(That \( \langle E, P \rangle \) is 2-maximizing is used for ‘only if’ again.) To get an idea of what goes on in cases (ii) and (iii), the reader may look at Theorem 2. If one of them applies, the “boundary” of one of \( U(A) \) and \( H(A) \) “generates” a bound as desired for both of them, unless the events outside of \( U(A) \) and \( H(A) \) according to case (iv) form a maximal antichain that is a bound as desired.

The previous theorems solve the problems formulated in Section 4 as explained in Section 5. This may still not be perfectly clear, so I give two examples in the ensuing corollary. Its first part deals with one class of instances of Question-Scheme 1*—the one Thomason seems to be interested exclusively, namely (up to isomorphism) just Question 1. The second part deals with Question 1+ which appears interesting to me.

43That is why they came to my mind when I looked for a characterization of \( \langle L, T \rangle \) being complete.—By [40, Lemma 2.23], for losets completeness is equivalent to Dedekind completeness, where the latter means that one of the members of every Dedekind cut contains a boundary of itself. Actually, these pairs together with the complement of the union of their members form just the triples that are instants of time in the sense of Walker’s construction according to [46].

44One might wonder whether an answer to a question “in terms of” \( \langle E, P \rangle \) may quantify over sets of events. However, the problem needs too much words of explaining, so I only give the following hints to my thoughts about it: (i) The elements of \( L \) are, admittedly, sets of events—but special ones. (ii) To characterize a second-order property of \( \langle L, T \rangle \) “in terms of” \( \langle E, P \rangle \), surely a second-order property of \( \langle E, P \rangle \) should be allowed (complexities should be allowed to match).

45Why did I put case (iv) so much more clumsy than this explanation? Answer: in order to “stay in terms of \( \langle E, P \rangle \)” . Otherwise, the theorem would not truly be an answer to Question 2, “in terms of” \( \langle E, P \rangle \). For the same (or respective) reason, I did not define \( e \in \exists_- \) by saying ‘the set of maximal antichains that contain e has a least element’ (as ‘first instant’, i.e.).—One could object that after having quantified over subsets of \( E \) there is no more point in such asceticism. I know, however, at least one possibility to counter this objection in some terms of complexity.—The formulation was inspired by the definition of a “complex open gap” in [17, p. 42].
Corollary 2. (a) \(\langle L, T \rangle\) is isomorphic to \(\langle I, < \rangle\) for some non-void open real interval \(I\) \((I = \mathbb{R},\ \text{e.g.})\) if and only if \(E\) neither has an \(SP\)-minimal element contained in \(\exists_-\) nor a \(PS\)-maximal element contained in \(\exists_+\), there is a countable subset \(D\) of \(E\) according to Theorem 3, and every \(A \subseteq E\) satisfies one of the cases of Theorem 4.

(b) \(\langle L, T \rangle\) is isomorphic to \(\langle I, < \rangle\) for some non-trivial real interval \(I\) if and only if there is a countable subset \(D\) of \(E\) according to Theorem 3, every \(A \subseteq E\) satisfies one of the cases of Theorem 4, and (if \(E\) has an \(SP\)-minimal element in \(\exists_-\) and a \(PS\)-maximal element in \(\exists_+)\) \(P \neq \emptyset\).

(Both (a) and (b) use countable choice and, for ‘only if’, that \(\langle E, P \rangle\) is 2-maximizing.)

8 Proofs.

Whereas so far I have tried to explain everything mentioned, the present section presumes some basic mathematical capabilities and, in places, some basic knowledge of axiomatic set theory as can be gained from, e.g., [27, Chapter 1] or [29].

8.1 “Constructivity” as opposed to choice principles.

This subsection starts elaborating the remarks beginning Section 7 on what the paper particularly assumes. I explain what I claimed there on avoidability of choice and similar principles as well as the loose use of terms like ‘constructive’. Neglecting a small amount of terminology used in parts of subsections 8.2 and 8.6, a reader used to apply the axiom of choice with the same ease as any other common set-theoretical principle may skip the subsection without loss.

The axiom of choice—AC for short—seems, in general, not to be questioned in nowadays’ mathematics and is considered part of standard (axiomatic) set theory.\textsuperscript{47} On the other hand, when it found broad mathematical attention for the first time nearly a hundred years ago,\textsuperscript{48} adversaries included Peano, Borel, Lebesgue, and Baire—see [28, pp. 103ff.] or, for more, [34]. Russell only temporarily was convinced of it ([1, p. 255]) and considered special assumptions on events to guarantee existence of instants without its use (see Subsection 8.2). In the meantime, there have, at least, always been mathematicians who liked to point it out when a proof used AC or similar non-constructive existence claims.\textsuperscript{49} Moreover, there is still some interest in

\textsuperscript{46}Otherwise \(P \neq \emptyset\) holds anyway.

\textsuperscript{47}Accordingly, [1, pp. 254ff.] and [46, p. 87] use it for a “modern” treatment of instants along Russell’s lines.

\textsuperscript{48}However, there was a ‘prehistory’ of a quarter of a century—cf. [34, pp. 5ff.].

\textsuperscript{49}At least, it seems to be common practice at Munich University to tell first-term students of mathematics that existence of bases for all vector spaces is due to Zorn’s
finding proofs only using constructive existence claims, even with an intuitionistic intent—here I am alluding to constructive mathematics, see, e.g., [8] or [45]. Subsections 8.1 and 8.2 address such interests.

In axiomatic set theory, everybody knows what “set theory without the axiom of choice” is. I designate this part of set theory by ZF.50 Subsections 8.1 and 8.2 tell something about which claims of the paper hold ‘by means of ZF only’. The latter phrase will return to stress this.

Whenever the paper claims that something holds “constructively” or that a proof is “constructive”, this just means that it is provable “by means of ZF only”. This terminology is, definitely, not correct considering what may be understood by constructivity in mathematics.51 I consider it justified, however, by the fact that purportedly “constructive” proofs avoid use of (ZF-)equivalents of AC ([41, p. xiv]) or of “weak versions” of it (other choice, maximality, or extension principles, some of which will be reviewed below) by presenting “definitions” the only free variables of which stand in place of things which exist by the hypothesis of the implication to be proved.52 This is to indicate how a formal proof could work.54 Thus, when the hypothesis of a theorem asserts existence of some \( x \) such that \ldots and claims respective existence of some \( y \) such that \ldots, a “definition” of such a \( y \) “in terms of \( x \)” suffices for “constructivity”—for holding “by means of ZF only”, i.e. there is no need to specify a respective \( x \) (by any “definition”).56

According to the beginning of Subsection 7, certain claims of the paper hold by means of ZF only. Like everywhere in set theory, the corresponding metamathematical claims will not be checked in a strict manner. The informal proofs only implicitly indicate their correctness. The specifically Lemma.

50 I prefer to consider ZF a purely formal theory, i.e., a certain set of formulas of some formal language of first-order predicate logic with identity, defined by a finite list of axioms and axiom-schemata; cf. [14, p. 13] or [27, p. xvi]. In the present paper, however, I cannot distinguish this formal theory from the body of informal theorems and proofs which could be formalized in “proper” ZF according to common practice (cf. [27, pp. 1f.]).

51 Constructive mathematics typically rejects the principle of tertium non datur which is not questioned in classical mathematics—cf. [45]. I will not check whether any proof invokes this principle.—The paper cannot address the question what ‘constructive’ ‘really’ means. To my knowledge, no precise answer is agreed upon by a majority of authors, cf. [8] and [45].

52 In spite of this “justification”, I will from now on prefer ‘by means of ZF only’ to ‘constructive’.

53 Cf. footnote 50 above.

54 My proofs of this kind may be acceptable for one or the other “constructive” mathematician—I hope so, but I have not checked this.

55 In the corresponding formal way, there typically is a defining class term containing no free variable but \( x \); cf. [29, p. 171].

56 Peter Schuster (the author of [45] and in this respect an expert) thinks that this feature of “relative constructivity” of some proofs may be accepted by many, though not by all exponents of constructive mathematics.
metamathematical aspects of the claims of the paper are addressed in an intermediately explicit manner in Subsection 8.2, in parts of Subsection 8.6, and in the rest of the present subsection as follows.

**Corollary 2** rests on (some of) **Facts 1, 1** and **1**, and so do similar claims I alluded to. My corresponding metamathematical claims rely on **Meta-Fact. Facts 1, 1** and **1**, hold by means of ZF only.

For those ‘facts’ (on the “object level”), I am referring to proofs to be found in the literature and to obvious variants of such proofs. For my corresponding metamathematical claim, one has to check that respective proofs can be carried out in ZF:

**Proof of Meta-Fact.** In each case, an isomorphism as wanted arises as a straightforward (indeed: unique) extension of an isomorphism of countable dense subchains which have no ends, obviously involving nothing more than ZF. That the construction of such a “core” isomorphism as well can be carried out by means of ZF only is to my knowledge most explicit in [36, p. 32f.] and in [5, p. 200].

**8.2 Existence of Russellian instants of time.**

This subsection reviews some metamathematical facts relating the basic assumption of the paper (according to the beginning of Section 7) that \( \langle E, P \rangle \) is 2-maximizing to AC and some of its consequences.

Russell, doubting AC, proposed to avoid it by \( E = \exists_\perp \) in [42] (like [49], see Section 7 above) and by ‘\( \langle E, P \rangle \) is 2-maximizing’ in [43] (see footnote 40 above) in order to guarantee existence of enough instants. These pains and

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57 Other expositions might appear—at first glance—to use the principle of dependent choices ([2, pp. 176f.], e.g.: cf. [29, pp. 168f.], [45]). Like in Subsection 8.2, however, countability allows to well-order the codomain and so to “define” a map—cf. Russell’s illustration by ‘pairs of shoes vs. socks’ (see [22, p. 351]). This is obvious for all the proofs I have seen, but the “recommended” proofs in [5] and in [36] are the only ones which make the definition explicit. The latter, however, might be supplemented by the hint that \( f^{-1} \) there is a second map which is constructed simultaneously with \( f \) and only in the end turns out to be the inverse map of the latter.

58 Consequences by means of ZF only, i.e., as explained in Subsection 8.1 above.

59 In [43], Russell discusses \( E = \exists_\perp \) again, as well \( E = \exists_\downarrow \) which serves the same purpose, viz., giving an instant to each event to be “at”; moreover \( S \setminus SPS \neq \emptyset \), which guarantees that there is any one instant at all (it is the negation of Thomason’s [46] condition corresponding to denseness of the Walker construction). Since he neither believes in AC nor in any of his own conditions, he remains sceptical about existence of instants. He might have been happy to know that “instants” may be constructed in other ways than as maximal antichains. Such alternatives are “generalized Dedekind-cuts” as in [46] or the entirely different, though surprisingly straightforward way of [17, Sections 7.5f.]. I propose this version of “generalized Dedekind-cuts”: For \( a \in E \), let \( h(a) := \{ e \in E \mid \text{not a PS e} \} \). Then \( K := \{ t \subseteq E \mid h(a) \subseteq t \text{ whenever } a \in t \} \) might be considered the set of instants, and an event \( a \) might be considered “at” ([42, Lecture IV]) an instant \( t \) if \( t \in h(t) \), but
8.2 Existence of Russelian instants of time.

possibilities of avoiding AC by first- or second-order constraints on structures of the special kind considered make “ZF only” more interesting to me than merely widespread curiosity about AC as mentioned in Subsection 8.1 above would have done.

In Subsection 8.6, I show that the characterizing condition of Theorem 3 is a similar condition on the particular structure under consideration which avoids AC in guaranteeing existence of enough instants. The present subsection deals with one further condition of this kind as well as with consequences of AC which, by contrast, generalize over all structures of the given kind.

Generalizing from \(\langle E, P \rangle\), a related set \(\langle X, R \rangle\) is an interval order whenever \(R\) is irreflexive and for all \(\langle x, x' \rangle, \langle y, y' \rangle \in R\) either \(x R y'\) or \(y R x'\). The reader will need no further help in generalizing to what an antichain in \(\langle X, R \rangle\) is. \(\langle X, R \rangle\) is maximizing if any antichain in \(\langle X, R \rangle\) extends to maximal one (equally clear). It is then, of course, \(n\)-maximizing for our earlier cases \(n = 0, 1, 2\). Let MAIO be the assumption that in every interval order there is a maximal antichain, and let MAIO’ be the assumption that every interval order is maximizing.

**Proposition 1.** By means of ZF only:

(a) MAIO’ is equivalent to MAIO.

(b) \(\langle X, R \rangle\) is maximizing provided \(X\) can be well-ordered.

The second part restates what Russell knew ([43] and as reported by [1, p. 254f.]). Accordingly, AC (being equivalent to the well-ordering principle) implies MAIO and thus (by the first part) MAIO’. By contrast to the proceeding presented here, MAIO’ derives from Zorn’s Lemma in an entirely elementary way (cf. [1, p. 255f.] deriving another variant of MAIO and MAIO’). The reason for my detour is: for philosophical reasons one might assume that the interval order \(\langle E, P \rangle\) has a countable underlying set \(E\); and then the following conclusion from the proposition might be of interest:

**Corollary A.** By means of ZF only: Each interval order the underlying set of which is countable is maximizing.

Remember the Henkin completeness proof for countable first-order languages goes without AC in contrast to the case of uncountable languages?—An application will follow in Subsection 8.6.\(^60\)

\(^60\) The reasoning carries over to other maximality principles, like the prime ideal theorem for Boolean algebras—being equivalent to the “ultrafilter theorem”—(see below) or like the existence of maximal cliques in a graph—which indeed generalizes MAIO.
Proof of Corollary A from Proposition 1. \( \omega \), being an ordinal ([27, p. 19]), is well-ordered by \( \in \) and by means of ZF only. If \( \phi \) is a one-to-one map from \( X \) into \( \omega \), let \( x \in X \) precede \( x' \in X \) iff their \( \phi \)-images behave accordingly; thus \( X \) is well-ordered.

Of course, Corollary A holds for every ordinal cardinal number (for every “aleph”, cf. Subsection 8.6 below) in place of \( \omega \); but, at the moment, I cannot imagine other infinite cardinals than \( \omega \) which for some philosophical reasons might be considered bounding the cardinality of the underlying set of events and, at the same time, is well-ordered by means of ZF only.\(^{61}\)

**Proof of Proposition 1.** Let \( \langle X, R \rangle \) be some interval order and \( Y \subseteq X \) an antichain.

(a) Let \( Y' := \{ y \in X \mid \{ y \} \cup Y \) is an antichain\} (so \( Y \subseteq Y' \)). Then \( \langle Y', R \rangle \) is an interval order which by assumption has a maximal antichain \( A \). By maximality of \( A \) and the definition of \( Y' \), \( Y \) must be a subset of \( A \). This proves one direction of the claim; the other follows trivially from the fact that \( \emptyset \) trivially is an antichain.

(b) If \( X \) can be well-ordered, there is a one-to-one map \( \phi \) from some ordinal \( \alpha \) onto \( X \setminus Y \). Define \( (Y'_\beta, Y_\beta) \) for \( \beta \in \alpha \) by \( Y'_0 := Y \), \( Y_\beta := Y'_\beta \cup \{ \phi(\beta) \} \) if this is an antichain and \( Y_\beta := Y'_\beta \) otherwise, \( Y'_{\beta+1} := Y'_{\beta} \) for \( \beta + 1 \in \alpha \), and \( Y'_\beta := \bigcup_{\gamma \in \beta} Y_\gamma \) for limit ordinals \( \beta \in \alpha \). Now \( Y \) is a subset of \( \bigcup_{\beta \in \alpha} Y_\beta \), and the latter is easily seen to be a maximal antichain.

Now I am going to consider two more ZF-consequences of AC which might seem relevant for \( \langle E, P \rangle \) 2-maximizing, in order to indicate that those assumptions about ‘maximizing’ are rather weak compared to AC.

Compare MAIO to Kurepa’s principle ‘in every poset [partially ordered set, i.e.] there is a maximal antichain’ which, assuming ZF, is equivalent to AC according to [15, pp. 61ff.]. Observe that an interval order \( \langle X, R \rangle \) is rendered a poset when \( R \) is replaced by its union with the diagonal of \( X \times X \). Since this would be a special kind of poset, I believe that Kurepa’s principle—whence neither AC—cannot be derived from ZF+MAIO alone.

By [15, pp. 128, 131ff.], the assumption BPI that in every Boolean algebra there is a prime ideal (which is, as is well known, derivable from ZF+AC and equivalent to existence of a maximal ultrafilter) does not, together with ZF only, imply AC. As, assuming ZF, MAIO is equivalent to the assumption that there is some somehow special ultrafilter in every Boolean algebra of some special kind (I cannot go into more detail here), I believe that, assuming nothing but ZF, BPI and MAIO do not imply each other. This belief implies that MAIO cannot be derived from ZF alone.

\(^{61}\)E.g., there is no well-ordering of the power set of \( \omega \) (equivalently: no well-ordering of the real numbers) by means of ZF only ([12, p. 138]).
8.2 Existence of Russelian instants of time.

By the previous content of the subsection I feel justified in assuming that \( \langle E, P \rangle \) is 2-maximizing. I.e., the reader is invited to choose his favourite from the justifications offered—maybe believing in AC (maybe the version most properly named so and vigorously advocated by Anderson and Gödel according to [1, p. 255]), maybe believing MAIO, maybe assuming \( E \) (a set of ‘[e]vents of which we are conscious’, [42, p. 121]) is countable and forgetting about all uncountable interval orders.

Some further notation introduced in Subsection 8.4 will indicate the role that assumptions on existence of instants play in the proofs.

Final remarks: Alternative constructions. (i) According to [46, Definition 2], Walker’s 1947 construction of instants of time and their ordering from \( \langle E, P \rangle \) yields a related set \( \langle L', T' \rangle \) where \( L' \) is the set of all triples \( \langle A, E \setminus (A \cup B), B \rangle \) such that \( A, B \) are non-void subsets of \( E \), \( a \ P \ b \) for all \( a \in A \) and all \( b \in B \), and finally for all \( c \in E \setminus (A \cup B) \) there are \( a \in A \) and \( b \in B \) such that \( a \ S \ c \ S \ b \); and where \( T' \) is the set of those ordered pairs \( \langle \langle A, C, B \rangle, \langle A', C', B' \rangle \rangle \) for which \( A \) is a proper subset of \( A' \). By [46, Proposition 3] (and because \( \langle E, P, SP, PS \rangle \) is an event ordering according to [46, Definition 1]), \( \langle L', T' \rangle \) is a complete loset.—Obviously, this ‘construction of time instants from events’ by means of ZF only (Russell may have been pleased to know).

(ii) Fishburn’s [17, pp. 23ff., 2.4.6] offers another ‘construction of time instants from events’ working by means of ZF only—however, without mentioning time.\(^{62}\) The same book as well as [16] present further variants using ZF only.

(iii) Now that I have mentioned three ways of ‘constructing time instants from events’, one of which I told not to be considered so by its author, it may be in order to explain what a ‘construction of time instants from events’ generally and from a purely mathematical point of view might be.—Following [42, Lecture IV], a ‘construction of time instants from events’ must not only supply some loset of “instants”, but must also tell for every event “at” which “instants” it occurs. To put this mathematically, fix some related set \( \langle L', T' \rangle \), and call \( \delta' \) a representation of \( \langle E, P \rangle \) in \( \langle L', T' \rangle \) if \( \delta' \) is a map assigning a non-void subset of \( L' \) to each event \( e \in E \) such that \( a \ P \ b \) iff, for all \( s \in \delta'(a) \) and all \( t \in \delta'(b) \), \( s \ T' \ t \). Philosophically, \( e \) now is “at” \( t \) (with respect to representation \( \delta' \)) iff \( t \in \delta'(e) \), i.e., \( \delta'(e) \) is the set of instants “at” which \( e \) occurs.—Now \( \delta(e) := \{ s \in L \mid e \in s \} \) defines a representation of \( \langle E, P \rangle \) in the Russelian \( \langle L, T \rangle \). More generally, a mathematical ‘construction of time instants from the events’ collected in \( E \) and ordered by \( P \) should just be a related set together with a representation of \( \langle E, P \rangle \) in that

\(^{62}\)Fishburn [17, pp. 41ff.] “constructs” Russell’s construction from his own. However, ‘constructively’ on [17, p. 45] must not be understood as if Fishburn had found a way to avoid AC in Russell’s construction—some part of AC is needed for the existence of ‘complex open gaps’. In particular, Fishburn does not show that Russelian instants exist at all in any interval order.
related set.\textsuperscript{63} Indeed, the proof of [17, Theorem 6, p. 29] provides a representation for every interval order in a loset as constructed by Fishburn [17, preceding there, pp. 23ff.]. Unfortunately however, the “tenure function” defined by [46, p. 94] is a representation of \((E, P)\) in the Walker loset if and only if \((E, P)\) is “dense” in Thomason’s sense.\textsuperscript{64} This might be considered an inadequacy of Walker’s construction. A different diagnosis is provided by Thomason [47, p. 52]: “Walker instants” just are not to “serve” as the “substance” of event durations (as the adequacy condition I introduced a few lines earlier demands), rather to separate events\textsuperscript{65} (Thomason’s respective diagnosis of the role of Russellian instants preceding there simply is wrong.)\textsuperscript{66}

8.3 Derived relations, witnesses, and linearity of time.

Define

\[ PSP := \{ (a, b) \mid a P c S d P b \text{ for some } c, d \} \]

This is, of course, some double composite of relations in two equivalent and well-known ways. The situation visualized in Figure 2 is just a case of \(a PSP b\); and \(PSP\) proves to be closely related with denseness. But it even is closely related with being an interval order (I am adding something more soon needed):

**Lemma 1.** (a) \(PSP \subseteq P\). (W)

(b) \(S\) is reflexive.

(c) \(P\) is transitive.

(d) \(SP\) and \(PS\) are irreflexive.

**Proof.** (a) If \(a P b S c P d\), then \(a P d\) follows from the definition of \(S\) and from (I) by propositional logic.

(b) This is an immediate consequence of the definition of \(S\) together with \(P\) being irreflexive.

\textsuperscript{63}I guess that in order to arrive at losets representing in this way, one has to apply “Occam’s razor”, viz., by imposing the restriction on such representing related sets that no proper restriction (now in the sense of Section 2 above) represents.

\textsuperscript{64}This sense conflicts with denseness as adopted also by Thomason for losets, since an interval order may well be a loset. Furthermore, Thomason’s “denseness” differs from “quasi-denseness” defined below.

\textsuperscript{65}To generalize what Thomason [47, p. 52] expects instants of time to do, I should like to suggest another notion of representation: Such a representation in a related set \(\langle L', T' \rangle\) should be a pair \((\lambda, \nu)\) of maps, each assigning some subset of \(L'\) to every event \(e \in E\) such that \(\lambda(e) \subseteq T' \nu(e)\), and such that \(a P b\) if \(\nu(a) \cap \lambda(b) \neq \emptyset\). The notation of \(!\) is introduced in Subsection 8.4 below

\textsuperscript{66}What [47, pp. 51ff.] sketches as a would-be ‘alternative description of Russell’s construction’ is, in fact, Fishburn’s [17, pp. 23ff., 29] construction, and [17, pp. 42ff.] demonstrates in what respects these two constructions, in general, differ.
(c) If \( a P b \), \( c P d \) and \( b = c \), then \( a P d \) follows from (W).

(d) Both \( e S c P e \) and \( e P c S e \) would contradict the definition of \( S \).

In fact, irreflexivity of \( P \) and (W) (Wiener’s [49] assumptions) could have been assumed instead of irreflexivity of \( P \) and (I).\(^ {67}\) To see this, firstly derive reflexivity of \( S \) from irreflexivity of \( P \), secondly transitivity of \( P \) from (W) and reflexivity of \( S \). Then, from \( a P b, c P d \) but not \( c P b \), similarly to the informal reasoning right after my introduction of (I) follows \( b P c \) or \( b S c \). \( a P d \) follows by transitivity of \( P \) in the first case and directly by (W) in the second case. Thus, the universal closure of (I) follows from (W) and irreflexivity of \( P \).

I am going to write ‘\( s T t \) with witnesses \( a,b \)’ if \( s T t \), \( a \in s \), \( b \in t \), and \( a P b \). By definition of \( T \), \( s T t \) is equivalent to there being witnesses according to this convention.

For better understanding some proofs below, it may be helpful to visualize witnesses of instants as horizontal strokes crossing vertical strokes representing these instants. E.g., Figure 3 shows witnesses \( a,c \) of \( s T t \) and witnesses \( d,b \) of \( t T u \). This situation forces \( c,d \) to overlap and \( a \) to wholly precede \( b \) (apart from vertical strokes, it is the same situation as in Figure 2).

Now I am ready for a proof of what I claimed in Section 3.

**Proposition 2.** \( \langle L,T \rangle \) is a non-void loset. (*0)

\( \langle \langle E,P \rangle \rangle \) is 0-maximizing’ has been invoked—fitting notation introduced in Subsection 8.4.) That \( \langle L,T \rangle \) is a loset is proved in Wiener’s [49, p. 445f.], but, alas, using Principia-notation. I am merely indicating what is going on to those who prefer to shun Principia-notation.

**Proof.** That \( L \) is non-void was decided in Subsection 8.2. \( s T s \) is excluded by the definitions of \( T \) and of an antichain—this is irreflexivity. If \( s,t \in L \)

\(^{67}\) Russell’s [43] starts with another equivalent set of conditions, viz., irreflexivity of \( P \) and transitivity of \( P \) and of \( SP \). Unfortunately, he did not fully recognize the significance of the transitivity of \( SP \)—see footnote 68 below.
The cardinality of the set $Z$ does not overlap some element $b$ of $t$. So $a P b$ or $b P a$, hence $s T t$ or $t T s$—this is connectedness. (As Wiener [49, p. 446] noted, both reasonings do not require any property of $P$ but being a binary relation on $E$.) Finally, let $s T t$ with witnesses $a, c$ and $t T u$ with witnesses $d, b$; by $c, d \in t$ we have $a P c S d P b$, so (W) yields $a P b$; this is transitivity of $T$, and, in the upshot, $T$ linearly orders $L$ (my original definition of linearity was redundant). (Note that irreflexivity of $P$ has not been used at all.) \[\Box\]

Russell [43] wrongly sketched a proof of why $T$ should be transitive assuming another set of assumptions on $\langle E, P \rangle$.\(^{68}\)

### 8.4 Further notation and Theorem 1.

In the rest of the paper, I will write ‘iff’ for ‘if and only if’, and $|Z|$ will, as usual, denote the cardinality of the set $Z$.\(^{69}\)

Concerning existence assumptions, I will moreover abbreviate ‘$(E, P)$ is 2-maximizing’ by ‘(*)’; as well, ‘(*)’ will indicate that (*) is invoked. ‘(*1)’ indicates that only ‘1-maximizing’ instead of ‘2-maximizing’ is used (cf. ‘(*0)’ with Proposition 2). ‘(*⇒)’ indicates that only the ‘if’ ’(*⇒)’ that only the ‘only if’ of an ‘iff’ is affected—analogues apply for ‘(1)’. ‘(*)’ will emphasize that a part of a statement has been invoked which does not involve any of these existence assumptions.

For $R \subseteq X \times Y$, let me define a number of “lifts”. For any set $Z$, let $P(Z)$ be its power set; and then

$$R! := \{ \langle x, Y' \rangle \in X \times P(Y) \mid \{x\} \times Y' \subseteq R \}.$$  

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\(^{68}\)This was observed by Thomason [46, p. 86] and overlooked by Anderson [1, p. 263, Note 6]. Russell claimed transitivity of $T$ derived from transitivity of $P$. The following counter-example refutes this claim. Let $E := \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and $P := \{\langle a_1, a_2\rangle, \langle a_1, a_3\rangle, \langle a_2, a_3\rangle, \langle b_1, b_2\rangle, \langle b_1, b_3\rangle, \langle b_2, b_3\rangle\}$. (A four-element $E$ would have sufficed to create the critical situation, but transitivity of $P$ would then have been trivial, and the example might have been less illustrative.) $P$ obviously is irreflexive and transitive. Now, among the elements of $L$ there will be $t := \{a_1, b_2\}$ and $t' := \{a_2, b_1\}$, and we have $t T t' T t$ (witnesses $a_1, a_2$, and $b_1, b_2$), but not $t T t$; so $T$ is not transitive.—Russell should instead have pointed at (W) and derived it using essentially his assumption of transitivity of $SP$ (cf. footnote 67 above). This would go as follows. Let a $P S b$. Assume for reductio not a $P b$. Then $b P a$ or $a S b$. (I am now going to write compositions of relations as [17, p. 1] does.) In the first case, $a S a P S b b P a$ by reflexivity of $S$ (which derives as for Lemma 1), so a $(SP)(SP)(SP) a$, and then a $SP a$ by transitivity of $SP$, contradicting irreflexivity of $SP$. (Note that irreflexivity of $SP$ as proved for Lemma 1 does not depend on any assumption on $P$, but is a consequence just of the definition of $S$ from $P$.) In the second case, $b S a P S b$, and this contradicts irreflexivity of $SP$ similarly to the first case.—Equivalence of the axiom set used in [43] (footnote 67 above) to each of the two two-element axiom sets suggested above now easily obtains, as transitivity of $SP$ is a simple consequence of (W).

\(^{69}\)Questioning AC raises some problems with cardinalities; then the definition of [29, p. 83, 2.2] is meant to apply.
\[ R_i := \{ (x, Y') \in X \times \mathcal{P}(Y) \mid (\{x\} \times Y') \cap R \neq \emptyset \}. \]

\(|R|\) and \(|r|\) are analogously defined as subsets of \(\mathcal{P}(X) \times Y\) such that \(X' \! R \ y\) means \(X' \times \{y\} \subseteq R\) and \(X' \! iR \ y\) writes \((X' \times \{y\}) \cap R \neq \emptyset\). By these definitions, \(|R|\) indicates that existential quantification occurs in the scope of universal quantification; but where parentheses do not matter I omit them, as with \(|R|\) or \(|r|\). As an example, now \(T = iP \cap (L \times L)\).

Furthermore, I let \(R_y := \{ x \in X \mid xRy \}\) for \(y \in Y\), and dually \(xR := \{ y \in Y \mid xRy \}\) for \(x \in X\). Thus, e.g., \(R_y Y' := \{ x \in X \mid (\{x\} \times Y') \cap R \neq \emptyset \}\) if \(Y' \subseteq Y\).

\[ \text{Lemma 2. Suppose } A \! P \! B, A' \subseteq A, \text{ and } B' \subseteq B. \text{ Then } V(A') \! T! V(B') \text{ and } V(A') \cap V(B') = \emptyset. \]

I will omit braces in the arguments of \(\Lambda\) and \(V\), so by \((*)\), \(\Lambda(c, d) \neq \emptyset\) whenever \(c S d\), and by \((*1)\), \(V(e) = \Lambda(e) \neq \emptyset\) for any event \(e\).

At this point, the proof of Theorem 1 \((|L| > 1 \text{ iff } P \neq \emptyset)\) seems convenient.

\[ \text{Proof of Theorem 1. Assume } P = \emptyset. \text{ Then } E \text{ is an antichain and, as it is the greatest subset of } E, \text{ a maximal one. Every other antichain properly} \]

\[ \text{extends to the antichain } E \text{ and thus is not maximal. So } L = \{E\} \text{ and therefore } |L| = 1. \]

\[ \text{Now assume there is } \langle a, b \rangle \in P. \text{ } V(a) (= \Lambda(a)) \text{ and } V(b) \text{ are both non-} \]

\[ \text{void } (*1) \text{ and, by Lemma 2, disjunct. Hence, } a \neq b \text{ and } |L| \geq |\{a, b\}| > 1. \]

For an explanation of the notion of \textit{duality}, cf. footnote 30 above. For a formal language, of course, one could easily \textit{prove} that any formal proof of some statement about \(\langle E, P \rangle\) (using perhaps formalizations of \((W)\) and the irreflexivity of \(P\) as axioms—note that they would be “self-dual”) dualizes to a proof of the dualized statement. Without a formal language, in the sequel I will omit proofs of dual statements just because readers could easily assure themselves of them by \textit{informally} dualizing the informal arguments presented explicitly (cf. [2, p. 13]).

\[ \text{Note that neither } R! \text{ nor } R! \text{ depends on } X \text{ or } Y. \text{ The analogous remarks apply to } !R \text{ and } !r \text{ below.} \]

\[ \text{Changing sides in dualizing now, of course, also applies to “lift marks” as defined above, } Ry \text{ becomes } yR, \text{ and } xr \text{ becomes } Rx. \text{ By contrast, } \Lambda \text{ and } V \text{ are “self-dual”}. \]
8.5 First and last instants; Theorem 2.

This subsection is concerned with the notions of having a first or, resp., a last instant, and proves Theorem 2.

A first instant of a subset \( A \) of \( E \) is a \( T \)-least element of \( V(A) \); dually, a last instant of \( A \) is a \( T \)-greatest element of \( V(A) \). By omitting braces of singletons, it is then clear what a first or last instant of some event is. As \( T \) is a linear order (and hence asymmetric), first and last instants are unique, of course.

The statements concerning first instants I present will always be duals of the statements concerning last instants. Therefore, I each time only prove one of these two cases.

**Lemma 3.** Let \( A \subseteq E \) and \( t \in L \).

(a) \( t \) is first [last] instant of \( A \) iff \( t \in V(A) \) and not \( V(A) \not< T t \) [not \( t T \not< V(A) \)].

(b) \( V(A) \not< T t \) iff \( A \not< T S \not< t \); and \( t T V(A) \) iff \( t \not< T \not< A \). (*)

**Proof.** (a) One direction follows from asymmetry—which follows from transitivity and irreflexivity—, the other from connectedness of linear orders.

(b) Assume there is \( s \in V(A) \) such that \( s T t \) with witnesses \( c, b \). Pick \( a \) from \( A \cap s \). Then \( s \in \Lambda(a, c) \) and therefore \( a S c P b \).—Now assume \( a SP b \) for some \( a \in A \) and some \( b \in t \). Then \( a S c P b \) for some \( c \). Choose \( s \) from \( \Lambda(a, c) \) (*). Then \( s T t \) by \( c P b \), and \( s \in \Lambda(a) \subseteq V(A) \).

In Section 6 I told that, as Russell found out,\(^\text{72}\) \( e \in \exists_- \) iff \( e \) has a first instant. One direction is proved in Wiener’s [49], the other in Russell’s [43]. For readers who find it hard to decipher the *Principia*-notation (and who prefer my notation), below I outline a proof in a different style.\(^\text{73}\) To this aim, for \( e \in E \) I define \( t_- (e) := e(S \setminus SP) \) and (dually) \( t_+ (e) := (S \setminus PS)e \).

**Lemma 4.** Suppose \( e \in E \).

(a) \( t_- (e) \) and \( t_+ (e) \) are antichains having \( e \) as an element.

(b) \( e \in \exists_- \) iff \( t_- (e) \) \((IP)\) \( e SP \); and \( e \in \exists_+ \) iff \( PSe \) \((P)\) \( t_+ (e) \).

(c) \( e \in \exists_- \) iff \( t_- (e) \in L \); and \( e \in \exists_+ \) iff \( t_+ (e) \in L \).

**Proof.** (a) \( e \in t_- (e) \cap t_+ (e) \) follows from reflectivity of \( S \) and irreflexivity of \( SP \) and \( PS \). Were \( a, b \in t_- (e) \) and \( a P b \), then \( e S a P b \) and hence \( e SP b \), contradicting \( b \in t_- (e) \).

(b) These are merely restatements of my original definitions of \( \exists_- \) and \( \exists_+ \) using the notation introduced in the meantime.

\(^\text{72}\)But cf. footnote 33 above for a reconsideration of the question of credits.

\(^\text{73}\)I hope the conventions I use are easier to keep in mind than the *Principia*-conventions—since their statement needed much less space than that of the *Principia*-symbols.
Suppose Proposition 4. Now: a singleton version of Lemma 3 with \( A \) and from the latter by Lemma 4 (b) follows \( e \) \( \text{SP} \) \( a \). By (b) the latter yields \( t_-(e) \) \( \text{iP} \) \( a \), and this contradicts the assumption \( a \) \( \text{SP} \) \( t_-(e) \).—Now assume \( t_-(e) \in L \) and \( e \) \( \text{SP} \) \( a \). By (b) the goal is to demonstrate that \( t_-(e) \) \( \text{iP} \) \( a \). For \text{reductio}, assume \( a \) \( (P \cup S)! \) \( t_-(e) \). But \( a \) \( P \) \( \in \) \( t_-(e) \) would imply \( e \) \( \text{SP} \) \( a \) \( b \), and, by transitivity of \( P \), \( e \) \( \text{SP} \) \( b \), contradicting \( b \in t_-(e) \). So \( a \) \( \text{SP} \) \( t_-(e) \), contradicting maximality of \( t_-(e) \) as an antichain, since by \( e \) \( \text{SP} \) \( a \) the event \( a \) is no element of \( t_-(e) \). \( \square \)

**Proposition 3.** Suppose \( e \in E \) and \( t \in L \).

(a) \( t \) is first instant of \( e \) iff \( t = t_-(e) \in L \); and \( t \) is last instant of \( e \) iff \( t = t_+(e) \in L \). (*1\( \Rightarrow \))

(b) \( e \in \exists_- \) iff \( e \) has a first instant; and \( e \in \exists_+ \) iff \( e \) has a last instant. (*1\( \Leftarrow \))

**Proof.** If \( t \) is first instant of \( e \), then, by Lemma 3 (*1),\(^{74} \) \( t \subseteq t_-(e) \). As \( t \) is a maximal antichain and \( t_-(e) \) is at least an antichain by Lemma 4 (a), even \( t = t_-(e) \). Now \( t_-(e) \in L \), and by Lemma 4 (c) \( e \in \exists_- \).—This was one direction of both parts of the proposition, but the remaining directions should be treated separately.

(a) Assume \( t_-(e) \in L \). By Lemma 4 (a) then even \( t_-(e) \in \Lambda(e) \). Let \( t' \neq t \) be another element of \( \Lambda(e) \). Thus, (as a maximal antichain cannot be a subset of a different one) there is \( b \in t' \setminus t_-(e) \). Then \( b \) \( S \) \( e \) and \( e \) \( \text{SP} \) \( b \), and from the latter by Lemma 4 (b) follows \( t_-(e) \) \( \text{iP} \) \( b \). Hence \( t_-(e) \) \( T \) \( t' \); and in the upshot, \( t_-(e) \) is first instant of \( e \).

(b) Assume \( e \in \exists_- \). Then by Lemma 4 (c), \( t_-(e) \in L \), and by the previous, \( t_-(e) \) is first instant of \( e \). \( \square \)

As the characterizing notions of Theorem 2 will go on to play an important role, I introduce symbols for them:

\[
\begin{align*}
\exists^* & := \{ A \subseteq E \mid \text{an SP-minimal element of } A \text{ is in } \exists_- \}; \\
\exists^*_+ & := \{ A \subseteq E \mid \text{a PS-maximal element of } A \text{ is in } \exists_+ \}.
\end{align*}
\]

Now:

**Proposition 4.** Suppose \( A \subseteq E \) and \( t \in L \).

(a) \( t \) is first instant of \( A \) iff \( t = t_-(e) \in L \) for some \( \text{SP-minimal element} \) \( e \) of \( A \); and \( t \) is last instant of \( A \) iff \( t = t_+(e) \in L \) for some \( \text{PS-maximal element} \) \( e \) of \( A \). (*1\( \Rightarrow \))

\(^{74}\)Using Lemma 3 for the proof so far, the proposition seems to rest on (*). However, a singleton version of Lemma 3 with \( A = \{e\} \) may be inserted after Lemma 4 which only needs (*1) and suffices for the present proposition. When \( e \) \( \text{SP} \) \( b \in t \), one then distinguishes whether \( e \in \exists_- \) or not. In the second case, there is some \( c \) such that \( e \) \( \text{SP} \) \( c \) \( PS \) \( e \) and \( c \) \( P \) \( b \) (cf. [43]) which just needs (*1) to be member of some \( s \in Tt \).
(b) $A \in \exists^*$ iff $A$ has a first instant; and $A \in \exists^*_+$ iff $A$ has a last instant. ($^\star \Leftarrow$)

Proof. (I am proceeding similarly as for Proposition 3.) If $t$ is first instant of $A$, then by Lemma 3 ($^\star$) there must be an $SP$-minimal element $a$ of $A$ such that $a \in t$. As $t$ cannot be a first instant of $A$ without being a first instant of $a$, by Proposition 3, $t$ must be $t_-(a) \in L$. Hence by Lemma 4 (c), $a \in \exists_-$ and, finally, $A \in \exists^*_-$.

—For the remaining:

(a) Assume $t_-(e) \in L$ for some $SP$-minimal element $e$ of $A$. By Lemma 4 (a) then $t_-(e) \in V(A)$. Let $t' \neq t$ be another element of $V(A)$. Thus, there is a $b \in (t' \cap A) \setminus t_-(e)$. $b P e$ by reflexivity of $S$ and $SP$-minimality of $e$ would contradict $b \in A$. Thus $e (S \cup P) b$, hence $e SP b$ and finally $t_-(e) T t'$ as in the proof of Proposition 3.

(b) Assume $A \in \exists^*_+$. So there is an $SP$-minimal element $e$ of $A$ in $\exists_-$. Then by Lemma 4 (c), $t_-(e) \in L$, and by the previous, $t_-(e)$ is first instant of $A$. □

Proof of Theorem 2. Theorem 2 is that instance of Proposition 4 (b) ($^\star \Leftarrow$) (with ‘iff’ reversed!) where $A = E$. □

8.6 Dense subsets; Theorem 3, Corollary 1.

For any subset $D$ of $E$ write

$$P_D := \{ (a, b) \mid a P c S d P b \text{ for some } c, d \in D \}.$$  

(Think of “$P$ as witnessed by $D$.”)

Call $D$ quasi-dense if for all $(a, b) \in P$ at the same time

(i) $a P_D b$ if $a \in \exists_+$ and $b \in \exists_-;

(ii) PSa \P_D b$ if $a \notin \exists_+;

(iii)$ (dually to the previous condition) $a P_D ! bSP$ if $b \notin \exists_-$.75

Existence of a countable quasi-dense subset of $E$ will in this subsection turn out to be one of the characterizing conditions. Being able to formulate this condition, I can now state what I earlier tried to announce about existence of instants.

Theorem A. By means of ZF only: If a countable subset of $E$ is quasi-dense, $(E, P)$ is maximizing.

75By [17, p. 5 Theorem 2], weakly as well as linearly ordered sets are a special case of interval orders. In this special case, denseness and quasi-denseness of subsets coincide, because then all elements just have one—first and last—instant, so only the first condition from the definition of quasi-denseness is relevant, and this one reduces to the usual condition defining denseness of a subset.
I need some lemma for this.

**Lemma 5.** Let $D$ be a quasi-dense subset of $E$, and let $\{a, b\} ! S! B \subseteq E$.

(a) If $a P c S c' P b$, then $\{c, c'\} ! S! B$.

(b) If $a \in \exists_+$ and $b \in \exists_-$, then there is $c \in D$ such that $a P c S! B$.

(c) If $a \notin \exists_+$, there is $c \in D$ such that $B ! S c P b$.

**Proof.** (a) Additionally to the hypotheses, assume $d \in B$, so $a S d S b$. $c P d$ would by $c P d S a P c$ and by (W) contradict irreflexivity of $P$. $d P c$, by $c' P b S d P c$ and (W), would contradict $c' S c$. Therefore, $c S d$, and in general, $c S! B$. A dual reasoning yields $c' S! B$.

(b) In the situation hypothesized there are $c, c' \in D$ such that $a P c S c' P b$, so the claim follows from (a).

(c) If $a \notin \exists_+$, assume $d \in B$, so $a S d S b$ and not $d PS a$, since otherwise $d P b$ by (W). By definition of $\exists_+$, there is $a' \in PSA$ such that not $a' P d$. Since not $d PS a$ and $P$ is transitive (Lemma 1), neither $d P a'$. Therefore, $a' S d S b$. Moreover, there are $c, c' \in D$ such that $a' P c' S c P b$, so the claim follows from (a). \[\square\]

**Proof of Theorem A.** Assume $D \subseteq E$ is countable and quasi-dense and $A$ is some antichain. Let $A' := \{e \in E \mid e S! A\}$. Since $A$ is an antichain, $A \subseteq A'$. If $A'$ is an antichain, it is a maximal one and we are ready. If not, there are $a, b \in A'$ such that $a P b$. By Lemma 5 and some supplement of dual reasoning, then, $D' := \{d \in D \mid d S! A\}$ turns out to be non-void. $\langle D', P \rangle$ is an interval order, where $D'$ is countable, and has, by Corollary A and by means of ZF only, a maximal antichain $D_0$. Let $A_0 := \{e \in E \mid e S! (A \cup D_0)\}$. Since $D_0 \subseteq D'$, we have $A ! S! D_0$, and definition of $A_0$ yields $A \subseteq A_0$. I have to show that $A_0$ is an antichain; definition of $A_0$ will then make clear that $A_0$ is a **maximal** antichain extending $A$.

To show that $A_0$ is an antichain, assume $a, b \in A_0$ and, for reductio, $a P b$. By Lemma 5 and some dual supplement, there is then some $c \in D$ such that $a P c$ or $c P b$ and $c S! (A \cup D_0)$. Therefore, $c \in D'$ and, since $D_0$ is a maximal antichain of $\langle D', P \rangle$, even $c \in D_0$. So $a S c S b$ by definition of $A_0$, contradicting $a P c$ or $c P b$. \[\square\]

I am now inserting some auxiliary facts which will be needed in the following.

**Lemma 6.** Suppose $a, b \in E$.

(a) $\Lambda(a) ! T! \Lambda(c, d) ! T! \Lambda(b)$ whenever $a P c S d P b$.

(b) Suppose, additionally, $t \in L$. If $V(a) ! T t$, then $a P i t$ and $a (PS) ! t$; and if $t T! V(b)$, then $t iP b$ and $t ! (SP) b$.\(^76\)

\(^76\)Parentheses with $PS$ and $SP$ assure that it is the relation composites that are “lifted”.
Proof. (a) follows immediately from the definitions of $\Lambda$ and $T$.

(b) Assume $V(a) ! T t$. Then $c P a$ for no $c \in t$ by asymmetry of $T$. $a S! t$ would imply $a \in t$, contradicting the assumption and irreflexivity of $T$. So there is some $c \in t \cap a P$ (i.e., $a P c \in t$), and then $a P c S! t$ which implies $a (PS)! t$. This proves the first statement; the other one follows dually.

Theorem 3 says that $\langle L, T \rangle$ is separable iff $E$ has a countable quasi-dense subset. More generally I can prove:

**Proposition 5.** Let $\kappa$ be an infinite cardinal.\(^{77}\) $L$ has a subset of quasi-dense

at most $\kappa$ being dense in $\langle L, T \rangle$ iff $E$ has a quasi-dense subset of cardinality

at most $\kappa$.\(^{78}\) (*)

---

\(^{77}\)The case of finiteness is somewhat trivial: a finite subset can be neither dense nor quasi-dense. (The 2002 versions of this paper were erroneous at this place.)

\(^{78}\)With regard to my discussion of what can be obtained “by means of ZF only” and in view of [29, V.1.8] (comparability of cardinalities), ‘at most’ is too vague and should be understood to mean that there is a one-to-one map from such a subset of $L$ or of $E$, resp., into some set cardinality of which is $\kappa$.

\(^{79}\)Theorem A naturally generalizes to quasi-dense $D \subseteq E$ when $\kappa$ is an aleph (see below) cardinal; so in this case (*) is entailed by the hypothesis.
that \( s \models t \models T \). That \( c_{s,t} \models S \models d_{t,u} \) obtains as before. \( e \models P \models c_{s,t} \models P \models c_{s,t} \) (by choices and by (W)) yields \( e \models P \models c_{s,t} \models P \models c_{s,t} \models S \models a \models P \models b \models d_{t,u} \models P \models b \). Thus, \( e \models P \models c_{s,t} \models S \models d_{t,u} \models P \models b \). (iii) works dually to the previous case.

Having discussed non-constructive existence assumptions guaranteeing the existence of maximal antichains, I should note that, given \( \kappa \), the proof uses some weaker version of the axiom of choice, viz., that there is a choice function on sets of cardinality \( \kappa \) ("\( \kappa \)-choice")—at least on such sets mentioned in the proof. Moreover, both ways round of the proof use \( \kappa^2 = \kappa \). This is entirely constructive if \( \kappa \) is a well-ordered (ordinal) cardinal, an "aleph"—see [29, p. 90, 2.32ff., pp. 96ff., 3.20ff.]. If your knowledge on cardinals derives from [27, pp. 27], you do not know of any other cardinals, and the only problem is \( \kappa \)-choice. Questioning AC however, \( \kappa \) may be something different, as defined, e.g., in [29, p. 83]. In this case, the statement that \( \kappa^2 = \kappa \) for all infinite cardinals \( \kappa \) is equivalent to the full axiom of choice—see [29, p. 164, 1.14].

Now the proposition is somewhat vague about \( \kappa \), and what it needs depends on how it is understood. Fortunately, Theorem 3 is a special case of the proposition looking much brighter:

**Proof of Theorem 3.** Theorem 3 is that special case of Proposition 5 where \( \kappa = \omega \) (*⇒).

By [29, p. 90, 2.32, 2.33 (ii), pp. 96, 3.20ff.], \( \omega^2 = \omega \) by means of ZF only. So the only "parts" of AC the theorem needs are the principle of countable choice and (*). While the latter assumption follows, by Theorem A, from the hypothesis that a countable quasi-dense subset exists, it is even needed for the second part of Proposition 5. Considering the facts reported in [29, pp. 167ff., 2.1, 2.3, 2.7] and [15, pp. 100ff., 155ff., 160ff.], I do not expect that the principle of countable choice and any maximizingness assumption (from (*0) up to MAIO) imply each other. Therefore, although the previous proposition made heavy use of AC, it still makes sense to track how little a part of it is needed for the central characterization results of the present paper.

Now I am turning to Corollary 1.

**Lemma 7.** If \( \langle L, T \rangle \) is dense, then

\[
P \cap (\exists_+ \times \exists_-) \subseteq PSP. \tag{D}
\]

**Proof.** Assume \( \langle L, T \rangle \) is dense. Consider \( \langle a, b \rangle \in P \cap (\exists_+ \times \exists_-) \). So by Lemma 4 and Proposition 3 (b) (¬*), \( t_+(a) \) is last instant of \( a \) and \( t_-(b) \) is first instant of \( b \). From \( a \models P \models b \) follows \( t_+(a) \models T \models t_-(b) \). So denseness of \( \langle L, T \rangle \) requires a \( t \in L \) such that \( t_+(a) \models T \models t_-(b) \). “Lastness” and “firstness” imply \( V(a) \models !T \models t \models V(b) \), so by Lemma 6 (b) there are \( c, d \in t \) such that \( a \models P \models c \models S \models d \models P \models b \).
Corollary B. Suppose $E$ is infinite. Then $L$ has a subset of at most the cardinality of $E$ dense in $(L,T)$ iff $(D)$ holds ($\Leftarrow$). So if $E$ is countable, $(L,T)$ is separable iff $(D)$ ($\Rightarrow$).

Proof. Assume $L$ has a subset of at most the cardinality of $E$ dense in $(L,T)$. Then, trivially, $(L,T)$ is dense, and $(D)$ holds by Lemma 7.

If $(D)$ holds, $E$ can be shown to be a quasi-dense subset of $E$ (*), so the other way statement follows from Proposition 5 (*). Indeed, $(D)$ is just condition $(i)$ of the definition of quasi-denseness in the case of $D = E$.

If $a \notin \exists_+$, consider $e \text{ PS } a$. There is an $e'$ such that $e P e' S a$, but because of $a \notin \exists_+$ we have $e' P S a$ again. $(W)$ and $a P b$ together yield $e P e' S e' P b$; so condition $(ii)$ of the definition of quasi-denseness holds; and condition $(iii)$ obtains dually.

Corollary 1 says that $(L,T)$ is dense iff $(D)$ holds.

Proof of Corollary 1. If $(L,T)$ is dense, $(D)$ again holds just by Lemma 7. Assuming $(D)$ for the other way round, the proof of Corollary B shows that $E$ is quasi-dense, and the proof of Proposition 5 shows how to construct a subset of $L$ dense in $(L,T)$—forget about its cardinality—from a quasi-dense subset of $E$ (*). Thus one sees that $(L,T)$ is dense. (One could as well have directly applied Corollary B having reasoned that $(D)$—if $P \neq \emptyset$—or events outside of $\exists_- \cap \exists_+$ force $E$ to be infinite, while $(L,T)$ trivially is dense in case of $P = \emptyset$ by Theorem 1.)

8.7 Completeness, Theorem 4.

Condition $(iv)$ of Theorem 4 needs some elucidation; to abridge it I therefore define

$$\exists^* := \{ \langle A,B \rangle \mid A,B \subseteq E; \ A \not\text{ PS } (E \setminus (A \cup B)); \text{ and } (E \setminus (A \cup B)) \not\text{ PS } \emptyset \}.$$ 

Theorem 4 can now be restated as

**Theorem 4'.** $(L,T)$ is complete iff every $D \subseteq E$ satisfies one of the following conditions: $(i)$ $D$ or $U(D)$ is empty; $(ii)$ $H(D) \in \exists^*$; $(iii)$ $U(D) \in \exists^*$; $(iv)$ $H(D) \exists^* U(D)$.

Its proof seems to require quite a number of steps.

**Lemma 8.** If $d$ is an SP-minimal element of $D \subseteq E$, then $P!D = Pd$; and if $d$ is a PS-maximal element of $D$, then $D!P = dP$.

Proof. Assume $d$ is an SP-minimal element of $D \subseteq E$. $P!D \subseteq Pd$ just follows from $d \in D$. As to the other way round, assume for reductio a $P \not\text{ PS } d$ and not a $P \not\text{ PS } e$ for some $e \in D$. Then $e P a$ or $e S a$. In the first case,
transitivity of $P$ (Lemma 1) yields $e P d$, and reflexivity of $S$ furthermore yields $e S e P d$, contradicting $SP$-minimality of $d$. In the second case, $e S a P d$ directly contradicts $SP$-minimality of $d$. The rest is duality. \qed

Recall that, by the notation introduced in this section, $U(D) = D!P$ and $H(D) = P!U(D)$ for every $D \subseteq E$. It may be helpful to compare the following lemma with propositions 3ff.

**Lemma 9.** Suppose $D \subseteq E$. Let $A := H(D)$, $B := U(D)$, and $C := E \setminus (A \cup B)$. Then:

(a) $P! A \subseteq A$, and $B \setminus P \subseteq B$;
(b) $C$ is an antichain;
(c) $A \nexists^* B$ iff $C \in L$;
(d) for all $t \in L$: $V(A)!T t T! V(B)$ iff $t = C \in L$;
(e) if $A \exists^* B$, then $A \notin \exists^*_+ B$ and $B \notin \exists^*_+ B$.

**Proof.** (a) follows easily from transitivity of $P$ and from the definitions of $A, B$.

(b) For reductio, assume there are $c, d \in C$ such that $c P d$. Since $d \notin B$, there is $a \in D$ such that not $a P d$, i.e., $d P a$ or $d S a$. In the first case, $d P! B$ by transitivity of $P$, so $d \in A$ contradicting $d \in C$. In the second case, for every $b \in B$ one gets $c P d S a P b$, so $c P b$ by (W), and in the upshot, $c P! B$. Therefore $c \in A$ contradicting $c \in C$.

(c) By (b), I just have to show that $A \exists^* B$ is equivalent to $C$ being maximal as an antichain. But being not maximal for $C$ is the same as $e S! C$ for some $e \in E \setminus C$, which directly contradicts $A \exists^* B$. For the other way round, if not $A \exists^* B$, then not $a P c$ for some $a \in A$ and every $c \in C$, or not $c P b$ for some $b \in B$ and every $c \in C$. Now, each of $c P a$ and $b P c$ by (a) would exclude $c \in C$, so, in fact, $a S! C$ or $C !S b$, and $C \cup \{a\}$ or $C \cup \{b\}$ is an antichain properly extending $C$.

(d) If $t \in L$ and $V(A)!T t T! V(B)$, then by Lemma 6 (b) and irreflexivity of $PS$ and $SP$ we have $t \subseteq C$. Since $t$ is a maximal antichain, $t = C$ follows from (b).—If, for the other way round, $C \in L$, then by (c) $A \exists^* B$, and from this $V(A)!T C T! V(B)$ follows easily.

(e) Assume $A \exists^* B$. I am going to show a little more than claimed, viz., that neither $A$ can have a $PS$-maximal element nor $B$ can have an $SP$-minimal element. For reductio, assume $B$ has an $SP$-minimal element $b$. Allowed to by $A \exists^* B$, pick $c$ from $C \cap Pb$. By Lemma 8 now $c P! B$, so $c \in A$ contradicting $c \in C$. Assuming $A$ has a $PS$-maximal element nearly dually leads to a contradiction—only recognize additionally $D \subseteq A$.\textsuperscript{80} \qed

**Lemma 10.** Let $M$ be a non-void subset of $L$ having no $T$-greatest element and satisfying $N := M \setminus T \neq \emptyset$. Let $D := P\iota M$, and let then $A, B, C$ be as in Lemma 9. Then:

\textsuperscript{80}I could have made better use of duality by showing $B = A \setminus P$, but then making sufficiently clear how duality would work here might be too difficult and expensive.
(a) \( M \subseteq V(D) \subseteq V(A) \);
(b) if \( t \) is an element of \( N \), but no \( T \)-least one, then \( t \in V(B) \);
(c) \( V(B) \subseteq N \);
(d) \( N = V(B) \), or \( N = \{ t \} \cup V(B) \), or \( N = \{ t \} \cup V(B) \) for some \( t \in V(A) \);
(e) if \( A \in \exists^* \), then \( N \cap V(A) \neq \emptyset \).

Proof. (a) Fix \( t \in M \). \( t \) is no \( T \)-greatest element of \( M \), so choose \( t' \in M \) such that \( t T t' \) with witnesses \( a, a' \). So \( a P a' \in t' \in M \), hence \( a P i t' \in M \) and \( a P i \ M \), i.e., \( a \in D \), and from \( a \in t \) follows \( t \in V(D) \). The inclusion \( V(D) \subseteq V(A) \) is just a consequence of \( D \subseteq A \).

(b) Fix \( t', t \in N \) such that \( t' T t \) with witnesses \( b', b \), and be \( a \in D \). So choose \( s \) from \( M \) having an element \( c \) such that \( a P c \). From \( t' \in N \) follows \( s T t' \) with witnesses \( d, d' \). Thus \( a P c S d \; d' S b' P b \), and from this \( a P b \) by double application of \( (W) \). In the upshot, \( D P ! b \), so \( b \in B \) and \( t \in V(B) \).

(c) By (a) and Lemma 2, we have \( M \setminus ! V(B) \), so \( V(B) \subseteq \emptyset \).

(d) Assume \( V(B) \neq N \neq \{ C \} \cup V(B) \). By (c), (b) and \( N \neq V(B) \), there is a \( T \)-least element \( t \) of \( N \) outside of \( V(B) \) such that \( N = \{ t \} \cup V(B) \). By \( T \)-leastness and being outside of \( V(B) \), \( t T ! V(B) \). By \( N \neq \{ C \} \cup V(B) \) we have \( t \neq C \), hence by Lemma 9 (d) \( not V(A) \setminus ! V(t) \). Thus \( t \in V(A) \) (and we are ready) or \( t T i V(A) \). In the latter case, however, by Lemma 3 (b) \((-\ast)\) there are \( d \in t \) and \( a \in A \) such that \( d PS a \). By the definition of \( A \), for every \( b \in B \) we have \( a P b \), therefore \( d PS a P b \), and, by \( (W) \), \( d P b \). Thus \( d \in A \) and \( t \in A(a) \subseteq V(A) \) as in the first case. (In fact, the latter case is impossible since \( (TiV(A)) \setminus !(Ti) \) \( M \) but I do not need this.)

(e) If \( A \in \exists^* \), by Proposition 4 (\(-\ast)\) \( V(A) \) has a \( T \)-greatest element \( t \). I have to show \( t \in N \). For \( \text{reductio} \), assume \( t \notin N \). Then \( t \in M \cup (Ti)M = TiM \), so there is \( t' \in M \) such that \( t T t' \). Since \( t \) is last instant of \( A \), \( V(A) \setminus ! T t' \), and by (a) \( M \setminus ! T t' \), contradicting irreflexivity of \( T \).

Proof of Theorem 4. Let \( M, N, D, A, B, C \) be as in Lemma 10. Since \( M \), having no \( T \)-greatest element, contains no upper bound of itself (with respect to \( T \)), \( N \) is the set of upper bounds of \( M \). Thus for completeness of \( \langle L, T \rangle \) I have to show that \( N \) has a \( T \)-least element if \( D \) satisfies one of the conditions (i) through (iv) of the theorem.\(^{81}\) (i) By Lemma 10 (a), since \( M \) is non-void, \( D \) cannot be empty. If, instead, \( B = \emptyset \), then \( V(B) = \emptyset \) as well, and by Lemma 10 (b) \( N \) has only one—\( T \)-least—element. (ii) If \( A \in \exists^* \), by Lemma 9 (c) and (c) \( C \notin L \), in particular \( C \notin N \). On the other hand, \( N \cap V(A) \neq \emptyset \) by Lemma 10 (e). By Lemma 2, \( V(A) \cap V(B) = \emptyset \), so \( N \neq V(B) \). Therefore, and as \( C \notin N \), by Lemma 10 (d), \( N = \{ t \} \cup V(B) \) for some \( t \in V(A) \). By Lemma 2 this \( t \) is \( T \)-least element of \( N \). (iii) If \( B \in \exists^* \),

\(^{81}\)The case of upper bounds suffices by [40, Exercise 2.20]; cf. the dual [24, p.14, Theorem 9].
like before $C \notin N$; furthermore, by Proposition 4 (\(-*\)), $V(B)$ has a $T$-least element. So by Lemma 10 (d), $N = \{t\} \cup V(B)$ for some $t \in V(A)$—in which case $t \in V(A)$ is a $T$-least element of $N$ as before—or $N = V(B)$—in which case the mentioned $T$-least element of $V(B)$ is $T$-least in $N$. (iv) If $A \exists^* B$, by Lemma 9 (c) and (d) together with Lemma 10 (a), $C \in N$. By Lemma 9 (d) together with irreflexivity of $T$, moreover $C \notin V(A) \cup V(B)$. So by Lemma 10 (d) $N = \{C\} \cup V(B)$, and again by Lemma 9 (d) $C$ is $T$-least element of $N$.

Now assume $(L,T)$ is complete, and let $D$ be some non-void subset of $E$ such that $B := U(D) \neq \emptyset$, $B \notin \exists^*_e$, and $A := H(D) \notin \exists^*_e$. I have to show that $A \exists^* B$. Because of $\emptyset \neq D \subseteq A$ we have $V(A) \neq \emptyset$, and from $B \neq \emptyset$ follows $V(B) \neq \emptyset$. By Lemma 2 we have $V(A) !T! V(B)$, so $V(A)$ has some upper bound with respect to $T$. By completeness of $(L,T)$, it must have a least upper bound $t$. According to Proposition 4 (\(*\)), $V(A)$ has no $T$-greatest and $V(B)$ has no $T$-least element. So $V(A) !T t! V(B)$. Now $A \exists^* B$ follows from Lemma 9 (d) and (c).

Corollary 2 only puts Theorems 1 through 4 as well as Facts 1 and 1 together and thus is clear. (For avoiding (\(*\)) in the ‘if’ direction, remember facts like Theorem A from Subsection 8.6. Need of countable choice is due to Theorem 3.)

9 Philosophical and historical remarks.

9.1 Russell on continuity of time and on conscious events required.

In [42, Lecture V], Russell acknowledged that time should be “continuous” in some sense. Though, he did not investigate continuity in the sense of being isomorphic to the real numbers or some real interval. Instead, he only discusses denseness of time—calling it ‘compactness’—and maintains this were the only ‘philosophically mattering’ among all the mathematical implications of continuity [42, Lecture V]. (Considering Principia Mathematica [48, *275] he must have been fully aware of what continuumlikeness is in the mathematical sense.) Accordingly, [42, Lecture IV] and [43] only present conditions on the ordering of events concerning denseness of time and nothing else coming as close to continuity as this. Russell completely ignores the question of separability—as the discussion of change and motion usually does—cf. Subsection 9.3. Neither, Dedekind-continuity is at stake.

I have no idea why Russell’s distinction of which ingredients of continuumlikeness matter philosophically and which do not should be right. Like Thomason, I have dealt with the remaining ingredients; but I cannot discuss here in which respect they really matter.

\footnote{Cf. footnote 37 above.}
Recall that the set of rational numbers, e.g., linearly ordered by $<$, forms, like that of the real numbers, a dense loset which, in contrast to the real numbers however, is not complete—so Russell seems to hold that a countable set of instants of time would “philosophically” do, in particular to explain motion to someone frightened by Zeno’s paradoxes.

Now Russell asked in [42, Lecture V]: ‘Is there, in actual empirical fact, any sufficient reason to believe the world of sense continuous?’ Since he was aware of no better characterization of event structures giving rise to density, he tried to decide the case of private time by recurring to (III).

Having replaced Russell’s Condition (III), which is sufficient but not necessary for density, by my characterization of density in Corollary 1, one might ask whether the situation has improved for Russell by that technical result.

Russell [42, Lecture V] concluded from his (III)\textsuperscript{83} that the number of events conscious to one being should be infinite in any finite period of time. Unfortunately, Russell seems to never have made explicit what a finite period of time could be.

Unfortunately enough, given any instants $s, t$ such that $s < t$, (infinitely iterated application of) density requires that infinitely many events start after $s$ and before $t$.\textsuperscript{84} But this follows directly from the definitions of density and of Russell’s instants; no characterization result is involved.

Russell goes on by saying ‘If this is to be the case in the world of one man’s sense-data, and if each sense-datum is to have not less than a certain finite temporal extension, it will be necessary to assume that we always have an infinite number of sense-data simultaneous with any given sense-datum.’

Here, the difficulty is to understand ‘not less than a certain finite temporal extension’. However, if all events simultaneous with one event $e$ are simultaneous with each other, they comprise a single instant, the only one at which $e$ is. In this case, $e$ could be no sense-datum as required by Russell, since it would last for one instant only and thus have no temporal extension at all. Therefore, for a given sense-datum $e$ there must be a preceding $b$, both simultaneous with $e$, and infinitely frequent application of (III) will verify Russell’s prophecy of an infinity of events simultaneous with $e$.

\textsuperscript{83}(III) is just (f) as in the long footnote of [42, Lecture IV]. At the passage to which I am referring here, Russell seems just to confuse it, intending to argue from (III) and not from the curious condition actually printed there.—Moreover, Russell’s discussion of conditions ‘required’ for something sometimes gives the impression that he had confused sufficiency and necessity of the condition for something (density, e.g.). At the presently discussed passage, however, there is no actual need for such an unfavorable interpretation. To make more clear what has already been said: Russell searches for a reason to believe in continuity, and he does not know anything else than (III) which could yield such a reason. Therefore, he discusses whether what (III) requires may be accepted.

\textsuperscript{84}The latter proposition can be made precise thus: there are infinitely many events each of which has an instant between $s$ and $t$.  

83, 84
Now, I must admit that the characterization by Corollary 1 does not yield any alleviation for Russell. Where infinitely many contemporaries of $e$ do not come in in the way of (III), they come in as members of infinite chains of contemporaries of contemporaries of $e$ which have no last or first instant ([43] explains the latter phenomenon).\(^{85}\)

One might rescue continuous private time by “possible” (private) events, which one could have arranged just to yield “possible” density. Kamp [23, p. 376] seems to think of the same solution.\(^{86}\)

Russell, however, felt ‘apparently forced to conclude that the space of sense-data is not continuous; but that does not prevent us from admitting that sense-data have parts which are not sense-data, and that the space of these parts may be continuous. The logical analysis we have been considering provides the apparatus for dealing with the various hypotheses, and the empirical decision between them is a problem for the psychologist.’ I wonder about the psychologist, but I let that aside. I conjecture that Russell’s last words on parts refer to Whitehead’s treatment of space as outlined by Russell’s [42, Lecture IV] before. Concerning time, Russell might as well have conceded that there are enough events giving rise to continuity without being sense-data.

Of course, density raises the same problem within the Walker framework according to Thomason’s [46]. Indeed, along these lines denseness of instants is equivalent to the condition that, if $a S b$, then $c P d$ for some events $c, d$ both of which are simultaneous with both $a$ and $b$. Applying this rule to $e S e$ yields infinitely many contemporaries of $e$.

In the remainder of [42, Lecture V], however, Russell seems to convince himself (and some readers may agree) that no serious philosophical problem is involved by the possibility that continuity could not appear already ‘in the world of one man’s sense-data’. Rather, mathematical and physical time in the received sense may be found by “logical analysis” (maybe using results of the present paper), and this suffices for solving any problem lingering in the literature.

9.2 Historical understandings of ‘continuum’.

‘The continuum’ seems to have always been a synonym for the set of the real numbers, maybe with some order or topological structure to make it a

\(^{85}\)E.g., let $e S a P b S e$. If $a \notin \exists_+$, then $a P c S d P b$ for some $c, d$ is not required for density by Corollary 1, while it would have been required by (III), and intrusion of an infinity of contemporaries of $e$ seems to be blocked. $a \notin \exists_+$, however, implies that the events overlapping $a$ and $e$ are not simultaneous with each other, since otherwise they would comprise a last instant of $a$. Rather, there are contemporaries $c_0$ and $d$ of $a$ and $e$ such that $c_0 P d$. Now $a \notin \exists_+$ furthermore forces a “never ending” situation $c_0 P c_1 P c_2 P \ldots c_n \ldots$ where the $c_n$ are contemporaries of $a$ and of $e$.

\(^{86}\)Kamp even indicates Russell could sometimes have pondered this solution, but does not provide a reference.
“line”; e.g., in Bolzano and Dedekind as criticized by Cantor in [10, p. 576] on the occasion mentioned below. This somewhat supports Thomason’s conception of time.

Note, however, that Thomason [46, p. 85] does not use the wording ‘the continuum’, but ‘a continuum’, e.g., ‘isomorphic to the real numbers’, also ‘continua’ as certain ‘orderings’—so it seems that, for Thomason, a continuum precisely is a linear order isomorphic to \( (\mathbb{R}, <) \).

On the other hand, subsets of certain spaces like, e.g., \( \mathbb{R}^n \) were called ‘continua’ in early topology provided they were compact (in \( \mathbb{R}^n \): closed and bounded, i.e.), (topologically) connected and had more than one element. So in \( \mathbb{R} \), e.g., the “continua” were just the non-trivial compact real intervals (by my terminology of Section 2). This notion of ‘continuum’ prevailed the work on an adequate general definition of ‘curve’ (cf. [39, p. 213], [38, p. 56]). Curves in \( \mathbb{R}^n \) were then one-dimensional continua in this sense (cf. [39, p. 235]). Even a loop (in particular, a “Jordan-curve”) was a “continuum”. Unfortunately, “the continuum” (being not compact) by this terminology is no “continuum”.

The previous definition of ‘continuum’, according to [32, p. 202], was for the first time suggested by the French mathematician C. Jordan in 1893, while the very first definition of ‘continuum’ (another one) was Cantor’s [10, 576]. Following Note 12 of the latter, the term ‘continuum’ had no generally accepted, definite meaning at that time.—To end this story: According to [38], the problem of a general definition of “curve” was solved in 1924 by the Russian topologist P. S. Urysohn. After Menger’s [33], I suppose, mathematical interest in the general notion of “curve” and, accordingly, in the general notion of “continuum” dissolved.\(^87\) ([38], published in 1957, was written for school teachers and amateur mathematicians.)

(The “Continuum Hypothesis”, by contrast, still has enjoyed broad attention in set theory and even in some other branches of mathematics; however, its debaters have, to my knowledge, not revealed which of the former actors takes the role of “the continuum” in this play—thus, its title notwithstanding, it cannot contribute to our story.)

The foregoing opposite conceptions of ‘continuum’ may be considered some justification for my choice at the end of Section 4, which generalizes both of them (if only subsets of \( \mathbb{R} \) are considered). Note that connectedness (as stipulated by Jordan’s definition) of the topological space naturally associated to a loset is equivalent to the latter being dense and complete [5, Chapter X Theorem 14]. Compactness just would add existence of a least and a greatest element [5, Chapter X Theorem 12].\(^88\) So the Jordan defi-

\(^87\) Curves are still indispensable in mathematics, but a general definition of them is dispensable. Different branches and authors define curves in their own way according to their particular needs, easily and without meditation—witness [25, p. 165], e.g.

\(^88\) Note the slight difference of the definition of ‘complete’ on [5, p. 6] compared to my definition in Section 5. ‘Complete’ in my sense is just ‘conditionally complete’ in [5,
9.3 Continuity vs. separability and completeness.

A number of authors (Russell in [42, Lecture V] being just one of them) discusses continuity of time in the context of change and, in particular, of motion. I cannot pursue this discussion here; I just should like to remark that this discussion seems to cover denseness and completeness only, ignoring separability.

Whether separability should be considered implicated by continuity or not, it should be discussed as a property of time and, thus, as a criterion to be imposed on an adequate reconstruction. This is easy to explain to those who take it for granted that time is dense and order-embeddable into \( \langle \mathbb{R}, < \rangle \). Separability of time is a necessary condition for that. To those who wonder why time should be dense and embeddable into \( \langle \mathbb{R}, < \rangle \), I should like to explain that if time is not separable, our measurement of it is deficient in some respect as follows. I try to sketch this explanation:

In principle, compactness means existence of all the least upper and greatest lower bounds that already exist by topological connectedness, as well as existence of a least upper and a greatest upper bound of the whole loset.

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\[ p. 114 \] Thus, compactness means existence of all the least upper and greatest lower bounds that already exist by topological connectedness, as well as existence of a least upper and a greatest upper bound of the whole loset.

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\[ \text{I wish I could explain it with better resources of various kinds. There is some literature at least touching on the subject which I have not read yet.} \]
we measure time going by by counting some adjacent intervals of time. Of course, we choose some chain of intervals which equal in “length” (as we think). If we need greater preciseness, we use shorter intervals. We like to be able to tell how many short intervals fit into one long interval. We think we can reach arbitrary precision by dividing any “former” unit interval into a certain number of parts. This means (I think) that for any two instants of time we can determine a chain of adjacent intervals such that the two instants happen to be in different intervals. Finally we think that for any such chain of adjacent intervals and for any two instants of time, there is a finite subchain of adjacent intervals such that one instant is in the first interval and the other in the last interval of that subchain (this corresponds to the archimedian axioms of measurement theory).—If time is not separable, however, measurement of time cannot work this way. Either some countable chain of adjacent intervals does not cover all of time, or there are instants we cannot distinguish by any enhancement of precision.

So much for separability. However, the matter is not even clear concerning denseness and completeness. We have seen (Subsection 9.1) that Russell thought the “philosophical” content of continuity were just denseness. There is another “extreme” in van Benthem’s [4, pp. 29ff.]

—where “continuity” explicitly just is completeness. I leave it open why denseness, completeness, or both should be considered ingredients of continuity. It has thus not been cleared up what “continuumlikeness” has to do with “continuity”.

9.4 Relative merits of Russell’s vs. Walker’s construction.

As mentioned in the beginning, Thomason [46, p. 95] tells: ‘Thus Walker’s theory offers, as Russell’s does not, a plausible explanation of time as a continuum.’ I object: Additionally to non-voidness of \(E\), irreflexivity of \(P\), and condition (I), continuumlikeness of Walker’s construction as well as of Russell’s construction imposes further restrictions on \(\langle E, P \rangle\). A ‘plausible explanation of time as a continuum’ requires an explanation of why these restrictions should obtain.\(^{92}\) For both constructions this is, I think, not easy. Concerning that part of a ‘plausible explanation of time as a continuum’ only which concerns conditions on \(\langle E, P \rangle\) necessary and sufficient for continuumlikeness of the constructed time, Walker’s construction does not fare better than Russell’s. This I hope to demonstrate by the ensuing two mathematical examples.

(i) \(\langle E, P \rangle\) might (up to isomorphism) consist of those open real intervals with rational boundaries, endowed with total precedence in the natural way. This is a case where Thomason’s characterizing conditions are satisfied and the construction \(\langle L', T' \rangle\) due to Walker/Thomason is isomorphic to \(\langle \mathbb{R}, \langle \rangle \rangle\)—so we have continuumlikeness. Concerning Russell’s construction

\(^{92}\)Cf. the programme according to Kamp [23, p. 381].
\(\langle L, T \rangle\), however, every rational number "doubles"—so \(\langle L, T \rangle\) is not dense and, consequently, not isomorphic to \(\langle \mathbb{R}, < \rangle\) (cf. [4, Theorem I.4.2.14] and [9, p. 246]).

(ii) Let \(\langle E, P \rangle\) (up to isomorphism) consist of all compact real intervals endowed with total precedence in the natural way. Then Thomason’s [46] denseness condition is not satisfied, every real number “doubles”, and \(\langle L', T' \rangle\) is very far from being dense. Russell’s \(\langle L, T \rangle\), on the other hand, is isomorphic to \(\langle \mathbb{R}, < \rangle\), i.e., continuumlike.

Admittedly, it may be considered an advantage for Walker’s construction that it yields completeness (“Dedekind-continuity”) “from the start”, i.e., without any restriction on \(\langle E, P \rangle\) (as opposed to Theorem 4).

Russell, being sceptical about the axiom of choice and about existence of instants along his own lines (cf. Subsection 8.1), might have been pleased about Walker’s construction which needs ZF only. He might however have criticized that Walker’s construction in general yields an instant “at” which an event occurs ([42, Lecture IV]) only under another special condition.

While the competition is undecided so far, a decision seems to come from category theory, as Thomason [47] maintains. However, I do not agree with all of that and hope to publish alternative views.

There are further rivals, some of which may be found in [16, 17] (cf. end of Subsection 8.2) and some of which will perhaps be published by myself.

9.5 “In metaphysicam”.

It is not my concern, as was Russell’s, that instants of time need philosophical saving from “metaphysicality” by construction from entities more close to our “senses”. I am interested in the distinction “anti-metaphysical” philosophers thought to have in mind, but I find other ways of explaining “empirical” or “cognitive content” of “theoretical” entities that philosophy has offered sufficiently promising, and I believe that “creation” of “entities” beyond ‘construction’ is what human (and other) beings really always have done and that this is just the way they work, in particular how they generate empirical hypotheses.44 My interest in the matter was for an essential part mathematical. Philosophically, however, I hoped to get some “feeling” for the empirical content of the postulate of “continuumlikeness” with respect to so many “quantities” in various areas of science.


94However, I can hardly elucidate the cognitive content of this statement. I have some ideas about what creation of an object and generate an empirical hypothesis by a subject could be, but I better do not try to tell.
9.6 Event relations beyond interval orders.

Thomason [46, 47], van Benthem [3, 4], and Kamp [23] advocate not to presume that the temporal relationships among events comprise an interval order. Their reasons and alternatives differ.

(i) In [46, p. 86], Thomason merely pretends that a structure

\[(E, P, SP, PS)\]

somehow comes more close to Russell’s presentation than a structure \((E, P)\). I do not see why. In [47, p. 50], Thomason remarks that in a category-theoretic setting \(SP\) and \(PS\) carry somewhat more information than \(P\) does alone—this is interesting, and I will just admit here that I remain somewhat perplexed about it.—Finally, [47] argues that an additional primitive relation between events (‘abuts’) must be considered.—I would like to reply, but this would need another paper.

(ii) Van Benthem seems as well to have strong reservations against reducing temporal relationships among events to just one relation. He uses an additional relation of ‘parthood’. Careful analysis of his publications seems to reveal that the only reason for these reservations is a reference [3, p. 7, ‘extremely doubtful’] to Kamp’s [23] criticism of the Russell/Wiener view of temporal relations of events.

(iii) Kamp [23, pp. 381–383] gives two reasons against the Russell/Wiener view of temporal relations of events. One reason is his linguistical objective of “discourse representation” where knowledge in general cannot decide precedence. The other reason plays the theme of vagueness where precedence cannot be decided from raw sense perception. To put forward a compromise: interval orders are a somewhat “idealized” picture of how events precede each other (or do not); at the same time, they present a somewhat “realistic” view. What some people perceive or (in another way) know about actual precedence is just another matter.

References


\(95\) See ‘must abandon postulate A7’ (p. 382) and ‘once again A7 must be abandoned’ (p. 383). I cannot explain in detail here the exact relation of A7 to interval orders. (I did this in a talk delivered in Amsterdam on February 13, 2002.) Just note that Kamp’s postulates A1–A7 are equivalent to the axioms of an interval order plus the definition of the derived relation of simultaneity.

\(96\) Suppose the sun is setting’, p. 383.

\(97\) Johan van Benthem agreed with me on that in the discussion following my talk in Amsterdam, February 13, 2002.
REFERENCES


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